

AP Calculus: l'Hôpital's Rule

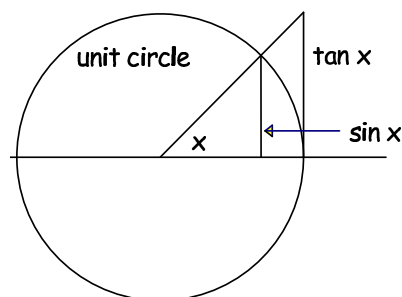
Very early in this course we encountered a fundamental “0/0 indeterminate form,” namely the limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}.$$

What makes this limit interesting is that both numerator and denominator tend to zero in the limit, making impossible a naive computation of the limiting ratio. Indeed, most interesting limits—such as those defining the derivative—are “indeterminate” in the sense that they are of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ where the numerator and denominator both tend to 0 (or to ∞). Students learn to compute the derivatives of trigonometric functions only after they have been shown that the limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

At the same time, you'll no doubt remember that the computation of this limit was geometrical in nature and involved an analysis of the diagram to the right.



The above limit is called a **0/0 indeterminate form** because the limits of both the numerator and denominator are 0.

You've seen many others; here are two more:

$$\lim_{x \rightarrow 3} \frac{2x^2 - 7x + 3}{x - 3} \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1}.$$

Note that in both cases the limits of the numerator and denominator are both 0. Thus, these limits, too, are 0/0 indeterminate forms.

While the above limits can be computed using purely algebraic methods, there is an alternative—and often quicker—method that can be used when algebra is combined with a little differential calculus.

Formally, a **0/0 indeterminate form** is a limit of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ where both $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. Assume, in addition, that f and g are both differentiable and that f' and g' are both continuous at $x = a$ (a very reasonable assumption, indeed!). Then we have

$$\begin{aligned}
\lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{\left(\frac{f(x)}{x-a}\right)}{\left(\frac{g(x)}{x-a}\right)} \\
&= \frac{\lim_{x \rightarrow a} \left(\frac{f(x)}{x-a}\right)}{\lim_{x \rightarrow a} \left(\frac{g(x)}{x-a}\right)} \\
&= \frac{f'(a)}{g'(a)} \\
&= \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)} \quad \text{(by continuity of the derivatives)}
\end{aligned}$$

This result we summarize as

l'Hôpital's Rule (0/0). Let f and g be functions differentiable on some interval containing $x = a$, that $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$, and assume that f' and g' are continuous at $x = a$. Then

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)}}.$$

As a simple illustration, watch this:

$$\lim_{x \rightarrow 3} \frac{2x^2 - 7x + 3}{x - 3} = \frac{\lim_{x \rightarrow 3} 4x - 7}{\lim_{x \rightarrow 3} 1} = 5,$$

which agrees with the answer obtained algebraically.

In a similar manner, one defines ∞/∞ indeterminate forms; these are treated as above, namely by differentiating numerator and denominator:

l'Hôpital's Rule (∞/∞). Let f and g be functions differentiable on some interval containing $x = a$, that $\lim_{x \rightarrow a} f(x) = \pm\infty = \lim_{x \rightarrow a} g(x)$, and assume that f' and g' are continuous at $x = a$. Then

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)}}.$$

There are other indeterminate forms as well: $0 \cdot \infty$, 1^∞ , and ∞^0 . These can be treated as indicated in the examples below.

EXAMPLE 1. Compute $\lim_{x \rightarrow 0^+} x^2 \ln x$. Note that this is a $0 \cdot \infty$ indeterminate form. It can easily be converted algebraically to an $\frac{\infty}{\infty}$ indeterminate form and handled as above:

$$\lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{(1/x^2)} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0^+} \frac{-x^2}{2} = 0.$$

Other indeterminate forms can be treated as in the following examples.

EXAMPLE 2. Compute $\lim_{x \rightarrow \infty} \left(1 - \frac{4}{x}\right)^x$. Here, if we set L equal to this limit (if it exists!), then we have, by continuity of the logarithm, that

$$\begin{aligned} \ln L &= \ln \lim_{x \rightarrow \infty} \left(1 - \frac{4}{x}\right)^x \\ &= \lim_{x \rightarrow \infty} \ln \left(1 - \frac{4}{x}\right)^x \\ &= \lim_{x \rightarrow \infty} x \ln \left(1 - \frac{4}{x}\right) \\ &= \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{4}{x}\right)}{1/x} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{4 / \left(x^2 \left(1 - \frac{4}{x}\right)\right)}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{-4}{\left(1 - \frac{4}{x}\right)} = -4 \end{aligned}$$

This says that $\ln L = -4$ which implies that $L = e^{-4}$.

EXAMPLE 3. This time, try $\lim_{\theta \rightarrow (\pi/2)^-} (\cos \theta)^{\cos \theta}$. The same trick applied above works here as well. Setting L to be this limit, we have

$$\begin{aligned} \ln L &= \ln \lim_{\theta \rightarrow (\pi/2)^-} (\cos \theta)^{\cos \theta} \\ &= \lim_{\theta \rightarrow (\pi/2)^-} \ln (\cos \theta)^{\cos \theta} \\ &= \lim_{\theta \rightarrow (\pi/2)^-} \cos \theta \ln \cos \theta \\ &= \lim_{\theta \rightarrow (\pi/2)^-} \frac{\ln \cos \theta}{1/\cos \theta} \\ &\stackrel{L'H}{=} \lim_{\theta \rightarrow (\pi/2)^-} \frac{\tan \theta}{\sec \theta \tan \theta} = 0. \end{aligned}$$

It follows that $\lim_{\theta \rightarrow (\pi/2)^-} (\cos \theta)^{\cos \theta} = 1$.

EXERCISES

1. For each improper integral below, compute its value (which might be $\pm\infty$ or determine that the integral does not exist.

(a) $\int_2^{\infty} \frac{dx}{\sqrt{x-2}}$

(b) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

(c) $\int_2^\infty \frac{dx}{\sqrt{x^2-4}}$

(d) $\int_0^1 \ln x \, dx$

2. Using l'Hôpital's rule if necessary, compute the limits indicated below:

(a) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{4x^3 - x - 3}$

(b) $\lim_{x \rightarrow 1} \frac{\cos(\pi x/2)}{\sqrt[3]{(x-1)^2}}$

(c) $\lim_{x \rightarrow \infty} \frac{2x^2 - 5x}{x^3 - x + 10}$

(d) $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\sin 4\theta}$

(e) $\lim_{\theta \rightarrow 0} \frac{\sin \theta^2}{\theta}$

(f) $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta}$

(g) $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\log_2 x}$

(h) $\lim_{x \rightarrow 0^+} (\ln x - \ln \sin x)$ (Hint: you need to convert this " $\infty - \infty$ " indeterminate

form to one of the forms discussed above!)

(i) $\lim_{x \rightarrow \infty} (\ln 2x - \ln(x+1))$.

(j) $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^x$

(k) $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x$

(l) $\lim_{x \rightarrow 1} x^{1/(x-1)}$

(m) $\lim_{x \rightarrow \infty} x^3 e^{-x}$

(n) $\lim_{x \rightarrow 0^+} x^a e^{-x}$, $a > 0$

(o) $\lim_{x \rightarrow 0^+} \ln x \ln(1-x)$

(p) $\lim_{x \rightarrow 1^-} \ln x \ln(1-x)$ (Are (o) and (p) really different?)