CHAPTER Five 5

Rational Expressions and Equations

In this chapter, we will examine rational expressions, which are fractions whose numerator and denominator are polynomials. We will learn to simplify rational expressions and how to add, subtract, multiply, and divide two rational expressions. Rational expressions have many applications, such as determining the maximum load that a wooden beam can support, finding the illumination from a light source, and solving work–rate problems.

Study Strategy **Test Taking** *To be successful in a math class, as well as understanding the material, you must be a good test taker. In this chapter, we will discuss the test-taking skills necessary for success in a math class.*

5.1 Rational Expressions and Functions

Objectives

1. **Evaluate rational expressions.**
2. **Find the values for which a rational expression is undefined.**
3. **Simplify rational expressions to lowest terms.**
4. **Identify factors in the numerator and denominator that are opposites.**
5. **Evaluate rational functions.**
6. **Find the domain of rational functions.**

A **rational expression** is a quotient of two polynomials, such as $\frac{x^2 + 15x + 44}{x^2 - 16}$. The denominator of a rational expression must not be zero, as division by zero is undefined. A major difference between rational expressions and linear or quadratic expressions is that a rational expression has one or more variables in the denominator.

Evaluating Rational Expressions

Objective 1 **Evaluate rational expressions.** We can evaluate a rational expression for a particular value of the variable just as we evaluated polynomials. We substitute the value for the variable in the expression and then simplify. When simplifying, we evaluate the numerator and denominator separately, and then simplify the resulting fraction.

EXAMPLE 1 Evaluate the rational expression $\frac{x^2 + 3x - 20}{x^2 - 5x - 8}$ for $x = -4$.

Solution

We begin by substituting -4 for x.

$$\frac{(-4)^2 + 3(-4) - 20}{(-4)^2 - 5(-4) - 8}$$ Substitute -4 for x.

$$= \frac{16 - 12 - 20}{16 + 20 - 8}$$ Simplify each term in the numerator and denominator. Note: $(-4)^2 = 16$

$$= \frac{-16}{28}$$ Simplify the numerator and denominator.

$$= -\frac{4}{7}$$ Simplify.

Quick Check 1
Evaluate the rational expression $\frac{x^2 - 6x - 1}{x^2 + 7x + 4}$ for $x = -2$.

Finding Values for Which a Rational Expression Is Undefined

Objective 2 **Find values for which a rational expression is undefined.** Rational expressions are undefined for values of the variable that cause the denominator to equal 0, as division by 0 is undefined. In general, to find the values for which a rational expression is undefined, we set the denominator equal to 0, ignoring the numerator, and solve the resulting equation.

EXAMPLE 2 Find the values for which the rational expression $\dfrac{8}{2x-3}$ is undefined.

Solution

We begin by setting the denominator, $2x - 3$, equal to 0 and then we solve for x.

$$\begin{aligned} 2x - 3 &= 0 && \text{Set the denominator equal to 0.} \\ 2x &= 3 && \text{Add 3 to both sides of the equation.} \\ x &= \frac{3}{2} && \text{Divide both sides by 2.} \end{aligned}$$

The expression $\dfrac{8}{2x-3}$ is undefined for $x = \dfrac{3}{2}$.

Quick Check 2
Find the values for which $\dfrac{6}{5x+9}$ is undefined.

EXAMPLE 3 Find the values for which $\dfrac{x^2+9x}{x^2-3x-40}$ is undefined.

Solution

We begin by setting the denominator $x^2 - 3x - 40$ equal to 0, ignoring the numerator. Notice that the resulting equation is quadratic and can be solved by factoring.

$$\begin{aligned} x^2 - 3x - 40 &= 0 && \text{Set the denominator equal to 0.} \\ (x+5)(x-8) &= 0 && \text{Factor } x^2 - 3x - 40. \\ x + 5 = 0 \quad \text{or} \quad x - 8 &= 0 && \text{Set each factor equal to 0.} \\ x = -5 \quad \text{or} \quad x &= 8 && \text{Solve.} \end{aligned}$$

The expression $\dfrac{x^2+9x}{x^2-3x-40}$ is undefined when $x = -5$ or $x = 8$.

Quick Check 3
Find the values for which $\dfrac{4x-7}{x^2-10x+21}$ is undefined.

Simplifying Rational Expressions to Lowest Terms

Objective 3 **Simplify rational expressions to lowest terms.** Rational expressions are often referred to as ***algebraic fractions***. As with numerical fractions, we will learn to simplify rational expressions to lowest terms. In later sections, we will learn to add, subtract, multiply, and divide rational expressions.

We simplified a numerical fraction to lowest terms by dividing out factors that were common to the numerator and denominator. For example, consider the fraction $\frac{30}{84}$. To simplify this fraction, we could begin by factoring the numerator and denominator.

$$\frac{30}{84} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 7}$$

The numerator and denominator have common factors of 2 and 3, which are divided out to simplify the fraction to lowest terms.

$$\frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 5}{\underset{1}{\cancel{2}} \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot 7} = \frac{5}{2 \cdot 7} \quad \text{or} \quad \frac{5}{14}$$

Simplifying Rational Expressions

To simplify a rational expression to lowest terms, we first factor the numerator and denominator completely. Then we divide out common factors in the numerator and denominator.

If P, Q, and R are polynomials, $Q \neq 0$, and $R \neq 0$, then $\frac{PR}{QR} = \frac{P}{Q}$.

EXAMPLE 4 Simplify the rational expression $\frac{15x^4}{6x^7}$. (Assume $x \neq 0$.)

Solution

This rational expression has a numerator and denominator that are monomials. In this case, we can simplify the expression using the properties of exponents developed in Chapter 4.

$$\frac{15x^4}{6x^7} = \frac{\overset{5}{\cancel{15}}x^4}{\underset{2}{\cancel{6}}x^7} \quad \text{Divide out the common factor 3.}$$

$$= \frac{5}{2x^3} \quad \text{Divide numerator and denominator by } x^4.$$

Quick Check 4
Simplify the rational expression $\frac{12x^8}{8x^2}$. (Assume $x \neq 0$.)

EXAMPLE 5 Simplify $\frac{(x-4)(x+6)}{(x+6)(x-1)(x-4)}$. (Assume the denominator is nonzero.)

Solution

In this example the numerator and denominator have already been factored. Notice that they share the common factors $x - 4$ and $x + 6$.

$$\frac{(x-4)(x+6)}{(x+6)(x-1)(x-4)} = \frac{\overset{1}{\cancel{(x-4)}}\overset{1}{\cancel{(x+6)}}}{\underset{1}{\cancel{(x+6)}}(x-1)\underset{1}{\cancel{(x-4)}}} \quad \text{Divide out common factors.}$$

$$= \frac{1}{x-1} \quad \text{Simplify.}$$

Notice that both factors were divided out of the numerator. Be careful to note that the numerator is equal to 1 in such a situation.

Quick Check 5
Simplify $\frac{(x-5)(x-3)(x-1)}{(x+1)(x-5)(x-3)}$. (Assume the denominator is nonzero.)

A Word of Caution If each factor in the numerator of a rational expression is divided out when simplifying the expression, be sure to write a 1 in the numerator.

EXAMPLE 6 Simplify $\frac{x^2 + 11x + 24}{x^2 + 4x - 32}$. (Assume the denominator is nonzero.)

Solution

The trinomials in the numerator and denominator must be factored before we can simplify this expression. (For a review of factoring techniques, you may refer to Sections 4.4 through 4.7.)

Quick Check 6
Simplify $\frac{x^2 - 3x - 18}{x^2 + 3x - 54}$. (Assume the denominator is nonzero.)

$$\frac{x^2+11x+24}{x^2+4x-32} = \frac{(x+3)(x+8)}{(x+8)(x-4)}$$ Factor numerator and denominator.

$$= \frac{(x+3)\cancel{(x+8)}^{1}}{\cancel{(x+8)}_{1}(x-4)}$$ Divide out common factors.

$$= \frac{(x+3)}{(x-4)}$$ Simplify.

A Word of Caution When simplifying a rational expression, be very careful to divide out only expressions that are common factors of the numerator and denominator. You cannot *reduce* individual terms in the numerator and denominator as in the following examples:

$$\frac{x+8}{x-6} \neq \frac{x+\cancel{8}^{4}}{x-\cancel{6}_{3}}$$

$$\frac{x^2-25}{x^2-36} \neq \frac{\cancel{x^2}^{1}-25}{\cancel{x^2}_{1}-36}$$

To avoid this, remember to factor the numerator and denominator completely before attempting to divide out common factors.

EXAMPLE 7 Simplify $\frac{2x^2-5x-12}{2x^2+2x-40}$. (Assume the denominator is nonzero.)

Solution

The trinomials in the numerator and denominator must be factored before we can simplify this expression. The numerator $2x^2-5x-12$ is a trinomial with a leading coefficient that is not equal to 1 and can be factored by grouping or by trial and error. (For a review of these factoring techniques, you may refer to Section 4.5.)

$$2x^2-5x-12 = (2x+3)(x-4)$$

The denominator $2x^2+2x-40$ has a common factor of 2 that must be factored out first.

$$\begin{aligned} 2x^2+2x-40 &= 2(x^2+x-20) \\ &= 2(x+5)(x-4) \end{aligned}$$

Now we can simplify the rational expression.

$$\frac{2x^2-5x-12}{2x^2+2x-40} = \frac{(2x+3)(x-4)}{2(x+5)(x-4)}$$ Factor numerator and denominator.

$$= \frac{(2x+3)\cancel{(x-4)}^{1}}{2(x+5)\cancel{(x-4)}_{1}}$$ Divide out common factors.

$$= \frac{2x+3}{2(x+5)}$$ Simplify.

***Quick Check* 7**

Simplify $\frac{3x^2-10x-8}{x^2-16}$. (Assume the denominator is nonzero.)

Identifying Factors in the Numerator and Denominator That Are Opposites

Objective 4 Identify factors in the numerator and denominator that are opposites. Two expressions of the form $a - b$ and $b - a$ are **opposites**. Subtraction in the opposite order produces the opposite result. Consider the expressions $a - b$ and $b - a$ when $a = 10$ and $b = 4$. In this case, $a - b = 10 - 4$ or 6, and $b - a = 4 - 10$ or -6. We can also see that $a - b$ is the opposite of $b - a$ by noting that $-(b - a) = -b + a = a - b$.

This is a useful result when we are simplifying rational expressions. The rational expression $\frac{a - b}{b - a}$ simplifies to -1, as any fraction whose numerator is the opposite of its denominator is equal to -1. If a rational expression has a factor in the numerator that is the opposite of a factor in the denominator, these two factors can be divided out to equal -1, as in the following example. We write the -1 in the numerator.

EXAMPLE 8 Simplify $\frac{x^2 - 7x + 12}{9 - x^2}$. (Assume that the denominator is nonzero.)

Solution

We begin by factoring the numerator and denominator completely.

$$\frac{x^2 - 7x + 12}{9 - x^2} = \frac{(x - 3)(x - 3)}{(3 + x)(3 - x)}$$ Factor the numerator and denominator.

$$= \frac{\overset{-1}{\cancel{(x - 3)}}(x - 3)}{(3 + x)\underset{1}{\cancel{(3 - x)}}}$$ Divide out the opposite factors.

$$= -\frac{x - 3}{3 + x}$$ Simplify, writing the negative sign in front of the fraction.

Quick Check 8

Simplify $\frac{7x - x^2}{x^2 - 13x + 42}$. (Assume the denominator is nonzero.)

A Word of Caution Two expressions of the form $a + b$ and $b + a$ are *not* opposites but are equal to each other. Addition in the opposite order produces the same result. When we divide two expressions of the form $a + b$ and $b + a$, the result is 1, not -1. For example, $\frac{x + 2}{2 + x} = 1$.

Rational Functions

Objective 5 Evaluate rational functions.

A **rational function** $r(x)$ is a function of the form $r(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$.

We begin our investigation of rational functions by learning to evaluate them.

EXAMPLE 9 For $r(x) = \dfrac{x^2 + 9x - 20}{x^2 - 3x + 16}$, find $r(6)$.

Solution

We begin by substituting 6 for x in the function.

$$\begin{aligned} r(6) &= \frac{(6)^2 + 9(6) - 20}{(6)^2 - 3(6) + 16} && \text{Substitute 6 for } x. \\ &= \frac{36 + 54 - 20}{36 - 18 + 16} && \text{Simplify each term in the numerator and denominator.} \\ &= \frac{70}{34} && \text{Simplify the numerator and denominator.} \\ &= \frac{35}{17} && \text{Simplify to lowest terms.} \end{aligned}$$

Quick Check 9

For $r(x) = \dfrac{x^2 - 5x + 6}{x^2 + 4x + 12}$, find $r(-6)$.

Finding the Domain of a Rational Function

Objective 6 **Find the domain of rational functions.** Rational functions differ from linear functions and quadratic functions in that the domain of a rational function is not always the set of real numbers. We have to exclude any value that causes the function to be undefined, namely any value for which the denominator is equal to 0. Suppose that the function $r(x)$ was undefined for $x = 6$. Then the domain of $r(x)$ is the set of all real numbers except 6. This can be expressed in interval notation as $(-\infty, 6) \cup (6, \infty)$, which is the union of the set of all real numbers that are less than 6 with the set of all real numbers that are greater than 6.

EXAMPLE 10 Find the domain of $r(x) = \dfrac{x^2 + 25}{x^2 - 14x + 45}$.

Solution

We begin by setting the denominator equal to 0 and solving for x.

$$\begin{aligned} x^2 - 14x + 45 &= 0 && \text{Set the denominator equal to 0.} \\ (x-5)(x-9) &= 0 && \text{Factor } x^2 - 14x + 45. \\ x - 5 = 0 \quad \text{or} \quad x - 9 &= 0 && \text{Set each factor equal to 0.} \\ x = 5 \quad \text{or} \quad x &= 9 && \text{Solve each equation.} \end{aligned}$$

The domain of the function is the set of all real numbers except 5 and 9. In interval notation this can be written as $(-\infty, 5) \cup (5, 9) \cup (9, \infty)$.

Quick Check 10

Find the domain of $r(x) = \dfrac{7x + 24}{x^2 + 6x - 16}$.

We must determine the domain of a rational function before simplifying it. If we divide a common factor out of the numerator before finding the domain of the function, we will miss a value of x that must be excluded from the domain of the function.

Building Your Study Strategy **Test Taking, 1** **Preparing Yourself** The first test-taking skill we will discuss is preparing yourself completely. As legendary basketball coach John Wooden once said, "Failing to prepare is preparing to fail." Start preparing for the exam well in advance; do not plan to study (or cram) on the night before the exam. To adequately prepare, you should know what the format of the exam will be. What topics will be covered on the test? How long will the test be?

Review your old homework assignments, notes, and note cards, spending more time on problems or concepts that you struggled with before. Work through the chapter review and chapter test in the text, to identify any areas of weakness for you. If you are having trouble with a particular type of problem, go back to that section in the text and review.

Make a practice test for yourself, and take it under test conditions without using your text or notes. Allow yourself the same amount of time that you will be allowed for the actual test, so you will know if you are working fast enough.

Exercises 5.1

FOR EXTRA HELP

MyMathLab

MathXL

Interactmath.com

MathXL Tutorials on CD

Video Lectures on CD

Addison-Wesley Math Tutor Center

Student's Solutions Manual

Vocabulary

1. A(n) __________ is a quotient of two polynomials.

2. Rational expressions are undefined for values of the variable that cause the __________ to equal 0.

3. A rational expression is said to be __________ if its numerator and denominator do not have any common factors.

4. Two expressions of the form $a - b$ and $b - a$ are __________ .

5. A __________ $r(x)$ is a function of the form $r(x) = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$.

6. The __________ of a rational function excludes all values for which the function is undefined.

Evaluate the rational expression for the given value of the variable.

7. $\dfrac{9}{x+8}$ for $x = 4$

8. $\dfrac{20}{3x-4}$ for $x = -8$

9. $\dfrac{5x-6}{x+10}$ for $x = -6$

10. $\dfrac{3x-10}{2x+4}$ for $x = 18$

11. $\dfrac{x^2-3x-2}{x^2+4x+3}$ for $x = 5$

12. $\dfrac{x^2-7x-18}{x^2+2x+3}$ for $x = 3$

13. $\dfrac{x^2+10x-9}{x^2+6}$ for $x = -2$

14. $\dfrac{x^2+5x+6}{x^2-3x-20}$ for $x = -7$

Find all values of the variable for which the rational expression is undefined.

15. $\dfrac{x+8}{x-6}$

16. $\dfrac{x-4}{2x-9}$

17. $\dfrac{x-9}{x(x+5)}$

18. $\dfrac{x+3}{(x+3)(x+7)}$

19. $\dfrac{3x+4}{x^2+15x+54}$

20. $\dfrac{x-10}{x^2-4x-32}$

21. $\dfrac{x}{x^2-25}$

22. $\dfrac{x^2+8}{x^2-3x}$

Simplify the given rational expression. (Assume that the denominator in each case is nonzero.)

23. $\dfrac{6x^3}{21x^7}$

24. $\dfrac{25x^9}{20x^{15}}$

25. $\dfrac{x-2}{(x+7)(x-2)}$

26. $\dfrac{(x-9)(x+8)}{x+8}$

27. $\dfrac{(x+6)(x-4)}{(x-4)(x-6)}$

28. $\dfrac{(x-7)(x-5)}{x(x-7)}$

29. $\dfrac{x^2+x-30}{x^2-7x+10}$

30. $\dfrac{x^2+6x}{x^2+10x+24}$

31. $\dfrac{x^2-11x+24}{x^2+x-72}$

32. $\dfrac{x^2+5x-14}{x^2+x-42}$

33. $\dfrac{x^2-12x+20}{x^2-4x+4}$

34. $\dfrac{x^2+3x-28}{x^2+14x+49}$

35. $\dfrac{x^2-36}{2x^2+15x+18}$

36. $\dfrac{3x^2-13x+4}{x^2-16}$

37. $\dfrac{4x^2-40x+96}{x^2-3x-4}$

38. $\dfrac{2x^2+14x+20}{3x^2-3x-90}$

39. $\dfrac{2x^2+5x-12}{x^2-3x-28}$

40. $\dfrac{3x^2-20x+12}{2x^2-7x-30}$

Determine whether the two given binomials are opposites or not.

41. $x+6$ and $6+x$

42. $x-8$ and $8-x$

43. $12-x$ and $x-12$

44. $5x-4$ and $4x-5$

45. $x+10$ and $-x-10$

46. $3x+5$ and $3x-5$

Simplify the given rational expression. (Assume that the denominator in each case is nonzero.)

47. $\dfrac{x-6}{(2+x)(6-x)}$

48. $\dfrac{(9+x)(9-x)}{x^2-2x-63}$

49. $\dfrac{x^2-11x+28}{16-x^2}$

50. $\dfrac{2x-x^2}{x^2+x-6}$

51. $\dfrac{36-x^2}{x^2+13x+42}$

52. $\dfrac{x^2-5x-14}{35-5x}$

Evaluate the given rational function.

53. $r(x) = \dfrac{3x - 6}{x^2 + 3x + 10}$, $r(-2)$

54. $r(x) = \dfrac{4x + 2}{x^2 + 5x - 14}$, $r(7)$

55. $r(x) = \dfrac{x^2 + 2x - 8}{x^2 - x - 8}$, $r(5)$

56. $r(x) = \dfrac{x^2 - 25}{x^2 + 10x + 24}$, $r(-4)$

57. $r(x) = \dfrac{x^3 - 2x^2 + 4x + 5}{3x^2 + 4x - 11}$, $r(3)$

58. $r(x) = \dfrac{x^3 - 6x^2 + 7x + 8}{x^3 + 5x^2 - 8x - 9}$, $r(2)$

Find the domain of the given rational function.

59. $r(x) = \dfrac{x^2 + 13x + 20}{x^2 - 6x}$

60. $r(x) = \dfrac{x^2 - 5x - 19}{x^2 + 4x + 3}$

61. $r(x) = \dfrac{x^2 - 7x - 10}{x^2 + 4x - 45}$

62. $r(x) = \dfrac{x^2 + 2x - 15}{x^2 - 25}$

63. $r(x) = \dfrac{x^2 + 48}{x^2 + 4x - 21}$

64. $r(x) = \dfrac{x^2 + 20x - 9}{x^2 - 17x + 72}$

Identify the given function as a linear function, a quadratic function, or a rational function.

65. $f(x) = x^2 - 8x + 16$

66. $f(x) = 5x + 15$

67. $f(x) = \dfrac{3x^2 + 2x + 12}{x}$

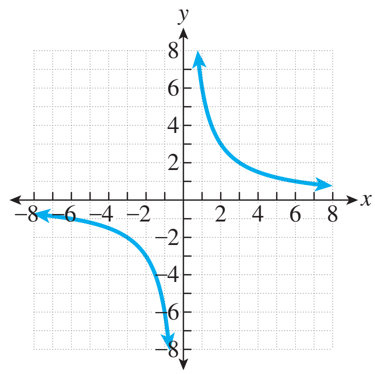
68. $f(x) = \dfrac{x^2 + 6x + 25}{2x^2 - 3x + 10}$

69. $f(x) = \dfrac{3}{4}x - 6$

70. $f(x) = \dfrac{2}{5}x^2 - \dfrac{3}{2}x + 5$

Use the given graph of a rational function r(x) to determine the following.

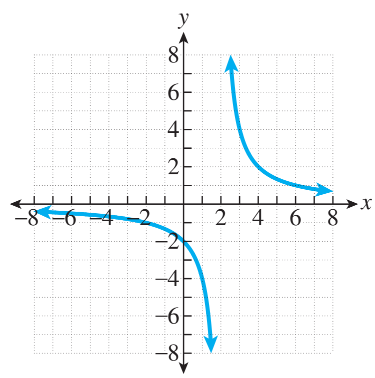
71.



a) Find $r(-1)$.

b) Find all values x such that $r(x) = -3$.

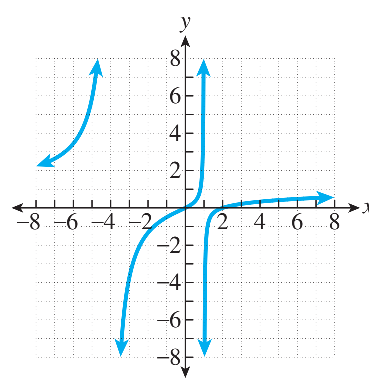
72.



a) Find $r(3)$.

b) Find all values x such that $r(x) = -2$.

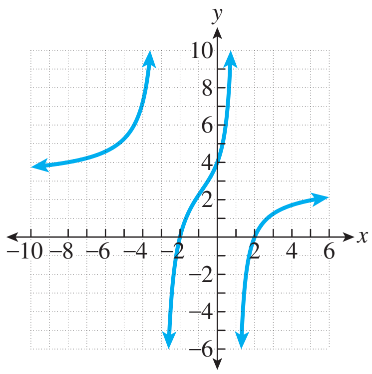
73.



a) Find $r(-5)$.

b) Find all values x such that $r(x) = 0$.

74.



a) Find $r(0)$.

b) Find all values x such that $r(x) = 0$.

Writing in Mathematics

Answer in complete sentences.

75. Explain how to find the values for which a rational expression is undefined.

76. Is the rational expression $\frac{(x+2)(x-7)}{(x-5)(x+2)}$ undefined for the value $x = -2$? Explain your answer.

77. Explain how to determine if two factors are opposites. Use examples to illustrate your explanation.

78. ***Solutions Manual**** Write a solutions manual page for the following problem:
Find the values for which the rational expression $\frac{x^2 + 3x - 28}{x^2 - 16}$ *is undefined.*

79. ***Newsletter**** Write a newsletter that explains how to simplify a rational expression.

***See Appendix B for details and sample answers.**

5.2 Multiplication and Division of Rational Expressions

Objectives

1. **Multiply two rational expressions.**
2. **Multiply two rational functions.**
3. **Divide a rational expression by another rational expression.**
4. **Divide a rational function by another rational function.**

Multiplying Rational Expressions

Objective 1 **Multiply two rational expressions.** In this section, we will learn how to multiply and divide rational expressions. Multiplying rational expressions is similar to multiplying numerical fractions. Suppose that we needed to multiply $\frac{4}{9} \cdot \frac{21}{10}$. Before multiplying, we can divide out factors common to one of the numerators and one of the denominators. For example, the first numerator (4) and the second denominator (10) have a common factor of 2 that can be divided out of each. The second numerator (21) and the first denominator (9) have a common factor of 3 that can be divided out as well.

$$\frac{4}{9} \cdot \frac{21}{10} = \frac{2 \cdot 2}{3 \cdot 3} \cdot \frac{3 \cdot 7}{2 \cdot 5} \quad \text{Factor each numerator and denominator.}$$

$$= \frac{\overset{1}{\cancel{2}} \cdot 2}{\underset{1}{\cancel{3}} \cdot 3} \cdot \frac{\overset{1}{\cancel{3}} \cdot 7}{\underset{1}{\cancel{2}} \cdot 5} \quad \text{Divide out common factors.}$$

$$= \frac{2 \cdot 7}{3 \cdot 5} \quad \text{Multiply remaining factors.}$$

$$= \frac{14}{15}$$

Multiplying Rational Expressions

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD} \qquad B \neq 0 \text{ and } D \neq 0$$

To multiply two rational expressions, we will begin by factoring each numerator and denominator completely. After dividing out factors common to a numerator and a denominator, we will express the product of the two rational expressions as a single rational expression, leaving the numerator and denominator in factored form.

Example 1 Multiply $\frac{x^2 - 6x - 27}{x^2 - 11x + 18} \cdot \frac{x^2 - 4}{x^2 + 9x + 18}$.

Solution

We begin by factoring each numerator and denominator. Then we proceed to divide out common factors.

$$\frac{x^2 - 6x - 27}{x^2 - 11x + 18} \cdot \frac{x^2 - 4}{x^2 + 9x + 18}$$

$$= \frac{(x-9)(x+3)}{(x-2)(x-9)} \cdot \frac{(x+2)(x-2)}{(x+3)(x+6)} \quad \text{Factor numerators and denominators completely.}$$

$$= \frac{\overset{1}{\cancel{(x-9)}}\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-2)}}\underset{1}{\cancel{(x-9)}}} \cdot \frac{(x+2)\overset{1}{\cancel{(x-2)}}}{\underset{1}{\cancel{(x+3)}}(x+6)}$$ Divide out common factors.

$$= \frac{x+2}{x+6}$$ Multiply.

Quick Check **1** **Multiply** $\frac{x^2+11x+18}{x^2+2x-15} \cdot \frac{x^2+13x+40}{x^2+5x-36}$.

EXAMPLE 2 Multiply $\frac{x^2-2x-35}{3x-x^2} \cdot \frac{x^2-13x+30}{2x^2-9x-35}$.

Solution

Again, we begin by completely factoring both numerators and denominators.

$$\frac{x^2-2x-35}{3x-x^2} \cdot \frac{x^2-13x+30}{2x^2-9x-35}$$

$$= \frac{(x+5)(x-7)}{x(3-x)} \cdot \frac{(x-3)(x-10)}{(2x+5)(x-7)}$$ Factor completely.

$$= \frac{(x+5)\overset{1}{\cancel{(x-7)}}}{x\underset{1}{\cancel{(3-x)}}} \cdot \frac{\overset{-1}{\cancel{(x-3)}}(x-10)}{(2x+5)\underset{1}{\cancel{(x-7)}}}$$ Divide out common factors. Notice that the factors $x-3$ and $3-x$ are opposites and divide out to equal -1.

$$= -\frac{(x+5)(x-10)}{x(2x+5)}$$ Multiply remaining factors. Write the negative sign in the numerator in front of the fraction.

Quick Check **2** **Multiply** $\frac{x^2-7x-30}{x^2-2x-8} \cdot \frac{x^2-9x+20}{100-x^2}$.

Here is a summary of the procedure for multiplying rational expressions.

Multiplying Rational Expressions

- Completely factor each numerator and each denominator.
- Divide out factors that are common to a numerator and a denominator, and divide out factors in a numerator and denominator that are opposites.
- Multiply the remaining factors, leaving the numerator and denominator in factored form.

Multiplying Rational Functions

Objective 2 **Multiply two rational functions.**

EXAMPLE 3 For $f(x) = \dfrac{x^2 + 4x - 12}{x^2 - 13x + 40}$ and $g(x) = \dfrac{x^2 - 6x - 16}{x^2 - 36}$, find $f(x) \cdot g(x)$.

Solution

We replace $f(x)$ and $g(x)$ by their formulas and proceed to multiply.

$$f(x) \cdot g(x) = \frac{x^2 + 4x - 12}{x^2 - 13x + 40} \cdot \frac{x^2 - 6x - 16}{x^2 - 36}$$
Replace $f(x)$ and $g(x)$ with their formulas.

$$= \frac{(x+6)(x-2)}{(x-5)(x-8)} \cdot \frac{(x+2)(x-8)}{(x+6)(x-6)}$$
Factor completely.

$$= \frac{\overset{1}{\cancel{(x+6)}}(x-2)}{(x-5)\underset{1}{\cancel{(x-8)}}} \cdot \frac{(x+2)\overset{1}{\cancel{(x-8)}}}{\underset{1}{\cancel{(x+6)}}(x-6)}$$
Divide out common factors.

$$= \frac{(x-2)(x+2)}{(x-5)(x-6)}$$
Multiply remaining factors.

Quick Check **3** For $f(x) = \dfrac{x + 8}{x^2 - x - 20}$ and $g(x) = \dfrac{x^2 - 15x + 50}{x^2 + 8x}$, find $f(x) \cdot g(x)$.

Dividing a Rational Expression by Another Rational Expression

Objective 3 **Divide a rational expression by another rational expression.**

Dividing a rational expression by another rational expression is similar to dividing a numerical fraction by another numerical fraction. We replace the divisor, which is the rational expression we are dividing by, by its reciprocal and then multiply. Replacing the divisor by its reciprocal is also called **inverting** the divisor.

Dividing Rational Expressions

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} \qquad B \neq 0,\ C \neq 0, \text{ and } D \neq 0$$

EXAMPLE 4 Divide $\dfrac{x^2 + 12x + 32}{x^2 - 11x + 30} \div \dfrac{x^2 + x - 12}{x^2 - 5x}$.

Solution

We begin by inverting the divisor and multiplying. In this example we must factor each numerator and denominator completely.

$$\frac{x^2 + 12x + 32}{x^2 - 11x + 30} \div \frac{x^2 + x - 12}{x^2 - 5x}$$

$$= \frac{x^2 + 12x + 32}{x^2 - 11x + 30} \cdot \frac{x^2 - 5x}{x^2 + x - 12}$$
Invert the divisor and multiply.

$$= \frac{(x+4)(x+8)}{(x-5)(x-6)} \cdot \frac{x(x-5)}{(x+4)(x-3)}$$ Factor completely.

$$= \frac{\overset{1}{\cancel{(x+4)}}(x+8)}{\underset{1}{\cancel{(x-5)}}(x-6)} \cdot \frac{x\overset{1}{\cancel{(x-5)}}}{\underset{1}{\cancel{(x+4)}}(x-3)}$$ Divide out common factors.

$$= \frac{x(x+8)}{(x-6)(x-3)}$$ Multiply remaining factors.

Quick Check 4 Divide $\frac{x^2-49}{x^2-14x+49} \div \frac{x^2+10x+21}{x^2-6x-7}$.

EXAMPLE 5 Divide $\frac{x^2-9x-10}{x^2-6x+8} \div \frac{x^2-1}{2-x}$.

Solution

We rewrite the problem as a multiplication problem by inverting the divisor. We then factor each numerator and denominator completely before dividing out common factors. (You may want to factor at the same time you invert the divisor.)

$$\frac{x^2-9x-10}{x^2-6x+8} \div \frac{x^2-1}{2-x}$$

$$= \frac{x^2-9x-10}{x^2-6x+8} \cdot \frac{2-x}{x^2-1}$$ Invert the divisor and multiply.

$$= \frac{(x-10)(x+1)}{(x-2)(x-4)} \cdot \frac{2-x}{(x+1)(x-1)}$$ Factor completely.

$$= \frac{(x-10)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x-2)}}(x-4)} \cdot \frac{\overset{-1}{\cancel{(2-x)}}}{\underset{1}{\cancel{(x+1)}}(x-1)}$$ Divide out common factors. Note that $2-x$ and $x-2$ are opposites.

$$= -\frac{x-10}{(x-4)(x-1)}$$ Multiply remaining factors.

Quick Check 5 Divide $\frac{x^2-4x-12}{x^2+13x+22} \div \frac{6-x}{x^2+15x+44}$.

Here is a summary of the procedure for dividing rational expressions.

Dividing Rational Expressions

- Invert the divisor and change the operation from division to multiplication.
- Completely factor each numerator and each denominator.
- Divide out factors that are common to a numerator and a denominator, and divide out factors in a numerator and denominator that are opposites.
- Multiply the remaining factors, leaving the numerator and denominator in factored form.

Dividing a Rational Function by Another Rational Function

Objective 4 **Divide a rational function by another rational function.**

EXAMPLE 6 For $f(x) = \dfrac{x^2 - 4x - 21}{x + 5}$ and $g(x) = x^2 + 8x + 15$, find $f(x) \div g(x)$.

Solution

Replace $f(x)$ and $g(x)$ by their formulas and divide. Treat $g(x)$ as a rational function with a denominator of 1. We can see that its reciprocal is $\dfrac{1}{x^2 + 8x + 15}$.

$$f(x) \div g(x) = \frac{x^2 - 4x - 21}{x + 5} \div (x^2 + 8x + 15)$$ Replace $f(x)$ and $g(x)$ by their formulas.

$$= \frac{x^2 - 4x - 21}{x + 5} \cdot \frac{1}{x^2 + 8x + 15}$$ Invert the divisor and multiply.

$$= \frac{(x - 7)(x + 3)}{x + 5} \cdot \frac{1}{(x + 3)(x + 5)}$$ Factor completely.

$$= \frac{(x - 7)\overset{1}{\cancel{(x + 3)}}}{x + 5} \cdot \frac{1}{\underset{1}{\cancel{(x + 3)}}(x + 5)}$$ Divide out common factors.

$$= \frac{x - 7}{(x + 5)^2}$$ Multiply remaining factors.

Quick Check 6

For $f(x) = \dfrac{x^2 + 16x + 64}{x^2 + x - 56}$ and $g(x) = x^2 + 7x - 8$, find $f(x) \div g(x)$.

Building Your Study Strategy **Test Taking, 2** **A Good Night's Sleep** Get a good night's sleep on the night before the exam. Tired students do not think as well as students who are rested. Some studies suggest that good-quality sleep on the two nights prior to an exam can have a positive effect on test scores. Also be sure to eat properly before an exam. Hungry students can be distracted during an exam.

EXERCISES 5.2

Vocabulary

1. To multiply two rational expressions, begin by __________ each numerator and denominator completely.

2. When multiplying two rational expressions, __________ out factors common to a numerator and a denominator.

3. When multiplying two rational expressions, once the numerator and the denominator do not share any common factors, express the product as a single rational expression, leaving the numerator and denominator in __________ form.

4. When dividing by a rational expression, replace the divisor by its __________ and then multiply.

Multiply.

5. $\dfrac{x+3}{(x-4)(x-6)} \cdot \dfrac{(x-4)(x+1)}{(x-1)(x+3)}$

6. $\dfrac{x+7}{x+8} \cdot \dfrac{(x+8)(x-2)}{(x+7)(x+2)}$

7. $\dfrac{x-1}{x-3} \cdot \dfrac{x^2+6x-27}{x^2-9x+8}$

8. $\dfrac{x^2+3x-10}{x^2+2x-35} \cdot \dfrac{x-5}{x-2}$

9. $\dfrac{x^2+4x}{x^2-2x-15} \cdot \dfrac{x^2-6x+5}{x^2+14x+40}$

10. $\dfrac{x^2+7x-18}{x^2-2x-48} \cdot \dfrac{x^2+5x-6}{x^2-2x}$

11. $\dfrac{8x-x^2}{x^2-5x+6} \cdot \dfrac{x^2+7x-30}{x^2+2x-80}$

12. $\dfrac{x^2-x-72}{x^2+12x+35} \cdot \dfrac{x^2+8x+7}{9x-x^2}$

13. $\dfrac{x^2+2x-8}{x^2-15x+54} \cdot \dfrac{81-x^2}{x^2-2x-24}$

14. $\dfrac{x^2+x-20}{64-x^2} \cdot \dfrac{x^2-15x+56}{x^2+9x+20}$

15. $\dfrac{x^2-18x+81}{x^2+10x+16} \cdot \dfrac{2x^2+13x-24}{x^2-5x-36}$

16. $\dfrac{x^2-4x-12}{x^2+3x-4} \cdot \dfrac{5x^2+23x+12}{x^2-14x+48}$

17. $\dfrac{x^2-144}{3x^2+10x+3} \cdot \dfrac{x^2+18x+45}{x^2-21x+108}$

18. $\dfrac{x^2-4x-77}{x^2-20x} \cdot \dfrac{x^2-15x-100}{x^2-6x-55}$

19. $\dfrac{2x^2-14x-36}{x^2-x-42} \cdot \dfrac{x^2+2x-24}{3x^2-12}$

20. $\dfrac{x^2+3x-40}{4x^2+36x+56} \cdot \dfrac{5x^2+25x+30}{x^2-4x-5}$

21. $\dfrac{2x^2+x-3}{x^2-12x+32} \cdot \dfrac{x^2-x-56}{2x^2+13x+15}$

22. $\dfrac{3x^2-4x}{x^2-3x-54} \cdot \dfrac{x^2+8x+12}{3x^2+2x-8}$

23. $\dfrac{x^3-5x^2+7x-35}{x^2+9x+18} \cdot \dfrac{x^2+5x+6}{x^2-6x+5}$

24. $\dfrac{x^3+6x^2-3x-18}{x^2-49} \cdot \dfrac{x^2+7x}{x^2-4x-60}$

25. $\dfrac{x^3-8}{x^2-5x-36} \cdot \dfrac{x^2-13x+36}{x^2-x-2}$

FOR EXTRA HELP

MyMathLab

MathXL

Interactmath.com

MathXL Tutorials on CD

Video Lectures on CD

Addison-Wesley Math Tutor Center

Student's Solutions Manual

26. $\dfrac{x^3 - 512}{x^2 + 4x + 4} \cdot \dfrac{x^2 + 10x + 16}{x^2 - 64}$

27. $\dfrac{x^3 + 1}{x^2 + 5x} \cdot \dfrac{x^2 + 3x - 10}{x^2 + 13x + 12}$

28. $\dfrac{x^3 + 1000}{x + 3} \cdot \dfrac{x^2 + 10x + 21}{x^2 + 5x - 50}$

For the given functions $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$.

29. $f(x) = \dfrac{7}{4 - x}, g(x) = \dfrac{x^2 - x - 12}{x^2 + 12x + 27}$

30. $f(x) = \dfrac{x^2 - 3x - 10}{x^2 + 2x - 3}, g(x) = \dfrac{x^2 + 8x - 9}{x^2 - 25}$

31. $f(x) = \dfrac{x^2 - 6x - 72}{x^2 - x - 6}, g(x) = \dfrac{x^2 - 6x + 9}{x^2 + 3x - 18}$

32. $f(x) = \dfrac{x^2 - 18x + 80}{x^2 + 2x - 63}, g(x) = \dfrac{x^2 + 13x + 36}{64 - x^2}$

33. $f(x) = \dfrac{x^2 + 5x - 24}{x^2 - 2x - 3}, g(x) = x^2 - 7x - 8$

34. $f(x) = x - 6, g(x) = \dfrac{x^2 + 5x - 66}{x^2 - 12x + 36}$

Find the missing numerator and denominator.

35. $\dfrac{(x + 5)(x - 2)}{(2x + 1)(x - 7)} \cdot \dfrac{?}{?} = \dfrac{(x + 5)}{(2x + 1)}$

36. $\dfrac{x^2 - 5x - 6}{x^2 + 15x + 54} \cdot \dfrac{?}{?} = \dfrac{(x + 1)(x + 5)}{(x + 6)^2}$

37. $\dfrac{x^2 - 10x + 16}{x^2 + 4x - 77} \cdot \dfrac{?}{?} = -\dfrac{x^2 - 11x + 24}{x^2 - 5x - 14}$

38. $\dfrac{x^2 + 4x}{x^2 - 81} \cdot \dfrac{?}{?} = \dfrac{x^2 - 6x}{x^2 - 7x - 18}$

Divide.

39. $\dfrac{(x + 7)(x - 5)}{(x + 2)(x + 15)} \div \dfrac{x + 7}{x + 15}$

40. $\dfrac{(x + 6)(x + 4)}{x - 5} \div \dfrac{x + 6}{(x - 2)(x - 5)}$

41. $\dfrac{x^2 + 15x + 56}{x^2 - 9x + 18} \div \dfrac{x^2 + 9x + 8}{x^2 - 7x + 6}$

42. $\dfrac{x^2 - 4x - 45}{x^2 + 4x + 4} \div \dfrac{x^2 - 16x + 63}{x^2 - 8x - 20}$

43. $\dfrac{x^2 + 16x + 63}{x^2 - 5x - 6} \div \dfrac{x^2 + 17x + 70}{x^2 + 11x + 10}$

44. $\dfrac{x^2 - 10x + 16}{x^2 + 4x - 5} \div \dfrac{x^2 - 5x - 24}{x^2 - 5x + 4}$

45. $\dfrac{x^2 + 14x + 45}{x^2 + 7x + 12} \div \dfrac{x^2 + 18x + 81}{3x^2 + 8x - 16}$

46. $\dfrac{x^2 - 12x + 35}{x^2 - 3x - 10} \div \dfrac{x^2 - 8x + 7}{x^2 + x - 2}$

47. $\dfrac{x^2 - 8x - 9}{x^2 - 10x + 9} \div \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$

48. $\dfrac{6x^2 + 7x + 1}{x^2 - 8x + 16} \div \dfrac{x^2 + 5x + 4}{x^2 - 16}$

49. $\dfrac{x^2 - 8x + 12}{x^2 - 8x + 15} \div \dfrac{4 - x^2}{x^2 + 5x - 50}$

50. $\dfrac{9 - x^2}{x^2 + 3x + 2} \div \dfrac{x^2 - 12x + 27}{x^2 + 10x + 9}$

51. $\dfrac{4x - x^2}{x^2 - 2x + 1} \div \dfrac{x^2 - 14x + 40}{x^2 + 6x - 7}$

52. $\dfrac{x - x^2}{x^2 + 11x + 30} \div \dfrac{x^2 - 10x + 9}{x^2 + 12x + 36}$

53. $\dfrac{3x^2 - 3x - 6}{2x^2 - 32x + 128} \div \dfrac{x^2 + 8x - 20}{x^2 - 64}$

54. $\dfrac{4x^2 + 24x + 20}{x^2 + 3x - 54} \div \dfrac{5x^2 + 105x + 100}{x^2 + 18x + 81}$

55. $\dfrac{x^2 + 3x}{2x^2 + 7x - 4} \div \dfrac{x^2 + 7x}{2x^2 + 3x - 2}$

56. $\dfrac{3x^2 - 13x - 10}{x^2 - 3x + 2} \div \dfrac{3x^2 + 11x + 6}{x^2 - 11x + 10}$

57. $\dfrac{x^3 - 5x^2 - 8x + 40}{x^2 + 3x} \div \dfrac{x^2 - x - 20}{x^2 + 5x + 6}$

58. $\dfrac{3x^3 + 7x^2 + 36x + 84}{x^2 - 16} \div \dfrac{3x + 7}{4x - x^2}$

59. $\dfrac{x^3 - 27}{x^2 - 8x + 12} \div \dfrac{9 - x^2}{x^2 + x - 42}$

60. $\dfrac{x^3 - 125}{x^2 - 4x - 5} \div \dfrac{x^2 + 15x + 54}{x^2 + 10x + 9}$

61. $\dfrac{x^3 + 216}{x + 3} \div \dfrac{x^2 + 21x + 90}{x^2 + 6x + 9}$

62. $\dfrac{x^3 + 64}{x^2 + 13x + 30} \div \dfrac{x^2 + 14x + 40}{x^2 + 10x + 21}$

For the given functions f(x) and g(x), find f(x) ÷ g(x).

63. $f(x) = \dfrac{x^2 + 13x + 42}{x^2 - 10x + 25}, g(x) = x + 6$

64. $f(x) = 3x + 15, g(x) = \dfrac{x^2 + 6x + 5}{x^2 + 17x + 72}$

65. $f(x) = \dfrac{x^2 + 6x + 9}{x^2 + 20x + 96}, g(x) = \dfrac{x^2 - 9}{x^2 + 8x}$

66. $f(x) = \dfrac{x^2 + 2x - 80}{x^2 + 10x + 25}, g(x) = \dfrac{x^2 + 16x + 60}{x^2 - 25}$

67. $f(x) = \dfrac{2x^2 + 3x}{x^2 - 16}, g(x) = \dfrac{2x^2 - 9x - 18}{x^2 - 19x + 60}$

68. $f(x) = \dfrac{x^2 + 20x + 100}{3x^2 - 10x + 3}, g(x) = \dfrac{x^2 - 100}{3x^2 + 11x - 4}$

Find the missing numerator and denominator.

69. $\dfrac{(x + 9)(x - 6)}{(x - 3)(x - 7)} \div \dfrac{?}{?} = \dfrac{x + 9}{x - 7}$

70. $\dfrac{x^2 - 5x + 4}{x^2 - 14x + 45} \div \dfrac{?}{?} = \dfrac{(x - 1)(x - 6)}{(x - 9)^2}$

71. $\dfrac{x^2 - 10x + 9}{x^2 + 7x} \div \dfrac{?}{?} = \dfrac{x^2 - 13x + 12}{x^2 + 11x + 28}$

72. $\dfrac{x^2 + 12x + 36}{x^2 - 100} \div \dfrac{?}{?} = \dfrac{x^2 - 36}{x^2 + 10x}$

Writing in Mathematics

Answer in complete sentences.

73. Explain the similarities between dividing numerical fractions and dividing rational expressions. Are there any differences?

74. Explain why the restrictions $B \neq 0$, $C \neq 0$, and $D \neq 0$ are necessary when dividing $\frac{A}{B} \div \frac{C}{D}$.

75. ***Solutions Manual**** Write a solutions manual page for the following problem:

Divide $\dfrac{x^2 - 6x - 27}{x^2 + 4x - 5} \div \dfrac{9x - x^2}{x^2 - 25}$.

76. ***Newsletter**** Write a newsletter that explains how to multiply two rational expressions.

***See Appendix B for details and sample answers.**

5.3 Addition and Subtraction of Rational Expressions

Objectives

1. **Add rational expressions with the same denominator.**
2. **Subtract rational expressions with the same denominator.**
3. **Add or subtract rational expressions with opposite denominators.**
4. **Find the least common denominator (LCD) of two or more rational expressions.**
5. **Add or subtract rational expressions with unlike denominators.**

Now that we have learned how to multiply and divide rational expressions, we move on to addition and subtraction. We know from our work with numerical fractions that two fractions must have the same denominator before we can add or subtract them. The same holds true for rational expressions. We will begin with rational expressions that already have the same denominator.

Adding Rational Expressions with the Same Denominator

Objective 1 **Add rational expressions with the same denominator.** To add fractions that have the same denominator, we add the numerators and place the result over the common denominator. We will follow the same procedure when adding two rational expressions. Of course, we should check that our result is in simplest terms.

Adding Rational Expressions with the Same Denominator

$$\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C} \qquad C \neq 0$$

Example 1 Add $\frac{9}{x+2} + \frac{7}{x+2}$.

Solution

These two fractions have the same denominator, so we add the two numerators and place the result over the common denominator $x + 2$.

$$\frac{9}{x+2} + \frac{7}{x+2} = \frac{9+7}{x+2}$$ Add numerators, placing the sum over the common denominator.

$$= \frac{16}{x+2}$$ Simplify the numerator.

The numerator and denominator do not have any common factors, so this is our final result. A common error is to attempt to divide a common factor out of 16 in the numerator and 2 in the denominator, but the number 2 is a term of the denominator and not a factor.

Quick Check 1

Add $\frac{3}{2x-5} + \frac{7}{2x-5}$.

EXAMPLE 2 Add $\dfrac{3x-8}{x^2-5x-24}+\dfrac{x+20}{x^2-5x-24}$.

Solution

The two denominators are the same, so we may add.

$$\frac{3x-8}{x^2-5x-24}+\frac{x+20}{x^2-5x-24}$$

$$=\frac{(3x-8)+(x+20)}{x^2-5x-24}\qquad\text{Add numerators.}$$

$$=\frac{4x+12}{x^2-5x-24}\qquad\text{Combine like terms.}$$

$$=\frac{4\cancel{(x+3)}^{1}}{(x-8)\cancel{(x+3)}_{1}}\qquad\text{Factor numerator and denominator and divide out the common factor.}$$

$$=\frac{4}{x-8}\qquad\text{Simplify.}$$

Quick Check **2** Add $\dfrac{2x+25}{x^2+10x+24}+\dfrac{3x-5}{x^2+10x+24}$.

EXAMPLE 3 Add $\dfrac{x^2+2x-9}{x^2-5x+4}+\dfrac{2x+4}{x^2-5x+4}$.

Solution

The denominators are the same, so we add the numerators and then simplify.

$$\frac{x^2+2x-9}{x^2-5x+4}+\frac{2x+4}{x^2-5x+4}$$

$$=\frac{(x^2+2x-9)+(2x+4)}{x^2-5x+4}\qquad\text{Add numerators.}$$

$$=\frac{x^2+4x-5}{x^2-5x+4}\qquad\text{Combine like terms.}$$

$$=\frac{(x+5)\cancel{(x-1)}^{1}}{\cancel{(x-1)}_{1}(x-4)}\qquad\text{Factor numerator and denominator and divide out the common factor.}$$

$$=\frac{x+5}{x-4}\qquad\text{Simplify.}$$

Quick Check **3** Add $\dfrac{x^2+4x+20}{x^2+7x-18}+\dfrac{8x+7}{x^2+7x-18}$.

Subtracting Rational Expressions with the Same Denominator

Objective 2 **Subtract rational expressions with the same denominator.**

Subtracting two rational expressions with the same denominator is just like adding them, except that we subtract the two numerators rather than adding them.

Subtracting Rational Expressions with the Same Denominator

$$\frac{A}{C} - \frac{B}{C} = \frac{A-B}{C} \qquad C \neq 0$$

EXAMPLE 4 Subtract $\frac{4x-3}{x-4} - \frac{x+9}{x-4}$.

Solution

The two denominators are the same, so we may subtract these two rational expressions. When the numerator of the second fraction has more than one term we must remember that we are subtracting the whole numerator and not just the first term. When we subtract the numerators and place the difference over the common denominator, it is a good idea to write each numerator in a set of parentheses. This will remind us to subtract each term in the second numerator.

$$\frac{4x-3}{x-4} - \frac{x+9}{x-4} = \frac{(4x-3)-(x+9)}{x-4} \qquad \text{Subtract numerators.}$$

$$= \frac{4x-3-x-9}{x-4} \qquad \text{Distribute.}$$

$$= \frac{3x-12}{x-4} \qquad \text{Combine like terms.}$$

$$= \frac{3\cancel{(x-4)}^{1}}{\cancel{x-4}_{1}} \qquad \text{Factor numerator and divide out the common factor.}$$

$$= 3 \qquad \text{Simplify.}$$

Notice that the denominator $x - 4$ was also a factor of the numerator, leaving a denominator of 1.

Quick Check 4

Subtract $\frac{3x+4}{x+8} - \frac{x-12}{x+8}$.

EXAMPLE 5 Subtract $\frac{2x^2-x-15}{6x-x^2} - \frac{x^2+3x-3}{6x-x^2}$.

Solution

The denominators are the same, so we can subtract these two rational expressions.

$$\frac{2x^2-x-15}{6x-x^2} - \frac{x^2+3x-3}{6x-x^2}$$

$$= \frac{(2x^2-x-15)-(x^2+3x-3)}{6x-x^2} \qquad \text{Subtract numerators.}$$

$$= \frac{2x^2 - x - 15 - x^2 - 3x + 3}{6x - x^2}$$ Change the sign of each term in the second set of parentheses by distributing -1.

$$= \frac{x^2 - 4x - 12}{6x - x^2}$$ Combine like terms.

$$= \frac{\overset{-1}{\cancel{(x - 6)}}(x + 2)}{x\underset{1}{\cancel{(6 - x)}}}$$ Factor numerator and denominator and divide out the common factor. The -1 results from the fact that $x - 6$ and $6 - x$ are opposites.

$$= -\frac{x + 2}{x}$$ Simplify.

Quick Check **5** Subtract $\frac{3x^2 - 6x - 14}{25 - x^2} - \frac{2x^2 - 8x + 21}{25 - x^2}$.

A Word of Caution When subtracting a rational expression whose numerator contains more than one term, be sure to subtract the entire numerator and not just the first term. One way to remember this is by placing the numerators inside sets of parentheses.

Adding or Subtracting Rational Expressions with Opposite Denominators

Objective 3 **Add or subtract rational expressions with opposite denominators.**

Consider the expression $\frac{10}{x - 2} + \frac{3}{2 - x}$. Are the two denominators the same? No, but they are opposites. We can rewrite the denominator $2 - x$ as its opposite $x - 2$ if we also rewrite the operation (addition) as its opposite (subtraction). In other words, we can rewrite the expression $\frac{10}{x - 2} + \frac{3}{2 - x}$ as $\frac{10}{x - 2} - \frac{3}{x - 2}$. Once the denominators are the same, we can subtract the numerators.

EXAMPLE 6 Add $\frac{8x}{3x - 15} + \frac{40}{15 - 3x}$.

Solution

The two denominators are opposites, so we may change the second denominator to $3x - 15$ by changing the operation from addition to subtraction.

$$\frac{8x}{3x - 15} + \frac{40}{15 - 3x}$$

$$= \frac{8x}{3x - 15} - \frac{40}{3x - 15}$$ Rewrite the second denominator as $3x - 15$ by changing the operation from addition to subtraction.

$$= \frac{8x - 40}{3x - 15}$$ Subtract the numerators.

$$= \frac{8\cancel{(x-5)}^{1}}{3\cancel{(x-5)}_{1}}$$ Factor the numerator and denominator and divide out the common factor.

$$= \frac{8}{3}$$ Simplify.

Quick Check 6

Add $\frac{2x}{3x - 18} + \frac{12}{18 - 3x}$.

Example 7 Subtract $\frac{x^2 + x + 5}{x^2 - 9} - \frac{6x + 7}{9 - x^2}$.

Solution

These two denominators are opposites, so we begin by rewriting the second rational expression in such a way that the two rational expressions have the same denominator.

$$\frac{x^2 + x + 5}{x^2 - 9} - \frac{6x + 7}{9 - x^2}$$

$$= \frac{x^2 + x + 5}{x^2 - 9} + \frac{6x + 7}{x^2 - 9}$$ Rewrite the second denominator as $x^2 - 9$ by changing the operation from subtraction to addition.

$$= \frac{(x^2 + x + 5) + (6x + 7)}{x^2 - 9}$$ Add the numerators.

$$= \frac{x^2 + 7x + 12}{x^2 - 9}$$ Combine like terms.

$$= \frac{\cancel{(x+3)}^{1}(x + 4)}{\cancel{(x+3)}_{1}(x - 3)}$$ Factor the numerator and divide out the common factor.

$$= \frac{x + 4}{x - 3}$$ Simplify.

Quick Check 7

Subtract

$\frac{x^2 - 7x + 10}{x^2 - 16} - \frac{x - 2}{16 - x^2}$.

The Least Common Denominator of Two or More Rational Expressions

Objective 4 **Find the least common denominator (LCD) of two or more rational expressions.** If two numerical fractions do not have the same denominator, we cannot add or subtract the fractions until we rewrite them as equivalent fractions with a common denominator. The same holds true for rational expressions with unlike denominators. We will begin by learning how to find the **least common denominator (LCD)** for two or more rational expressions.

Finding the LCD of Two Rational Expressions

Begin by completely factoring each denominator, and then identify each expression that is a factor of one or both denominators. The LCD is equal to the product of these factors.

If an expression is a repeated factor of one or more of the denominators, then we repeat it as a factor in the LCD as well. The exponent used for this factor is equal to the greatest power that the factor is raised to in any one denominator.

EXAMPLE 8 Find the LCD of $\frac{3}{10a^2b}$ and $\frac{5}{12a^5}$.

Solution

We begin with the coefficients 10 and 12. The smallest number that both divide into evenly is 60. Moving on to variable factors in the denominator, we see that the variables a and b are factors of one or both denominators. Note that the variable a is raised to the fifth power in the second denominator, so the LCD must contain a factor of a^5. The LCD is $60a^5b$.

***Quick Check* 8** **Find the LCD of $\frac{7}{6x^3y^2}$ and $\frac{3}{8x^6y}$.**

EXAMPLE 9 Find the LCD of $\frac{x+7}{x^2+x-30}$ and $\frac{x-4}{x^2-36}$.

Solution

We begin by factoring each denominator.

$$\frac{x+7}{x^2+x-30} = \frac{x+7}{(x+6)(x-5)} \qquad \frac{x-4}{x^2-36} = \frac{x-4}{(x+6)(x-6)}$$

The factors in the denominators are $x+6$, $x-5$, and $x-6$. Since no expression is repeated as a factor in any one denominator, the LCD is $(x+6)(x-5)(x-6)$.

***Quick Check* 9**

Find the LCD of $\frac{x+5}{x^2+11x+24}$ and $\frac{x-3}{x^2-3x-18}$.

EXAMPLE 10 Find the LCD of $\frac{x+9}{x^2+4x-21}$ and $\frac{x}{x^2-6x+9}$.

Solution

Again, we begin by factoring each denominator.

$$\frac{x+9}{x^2+4x-21} = \frac{x+9}{(x+7)(x-3)} \qquad \frac{x}{x^2-6x+9} = \frac{x}{(x-3)(x-3)}$$

The two expressions that are factors are $x+7$ and $x-3$; the factor $x-3$ is repeated twice in the second denominator. So the LCD must have $x-3$ as a factor twice as well. The LCD is $(x+7)(x-3)(x-3)$ or $(x+7)(x-3)^2$.

***Quick Check* 10**

Find the LCD of $\frac{2x+5}{x^2-36}$ and $\frac{x-9}{x^2+12x+36}$.

Adding or Subtracting Rational Expressions with Unlike Denominators

Objective 5 **Add or subtract rational expressions with the same denominators.** To add or subtract two rational expressions that do not have the same denominator, we begin by finding the LCD. We then convert each rational expression to an equivalent rational expression that has the LCD as its denominator. We can then add or subtract as we did in the previous section. As always, we should attempt to simplify the resulting rational expression.

EXAMPLE 11 Add $\frac{5}{x+4} + \frac{3}{x+6}$.

Solution

The two denominators are not the same, so we begin by finding the LCD for these two rational expressions. Each denominator has a single factor, and the LCD is the product of these two denominators. The LCD is $(x+4)(x+6)$. We will multiply $\frac{5}{x+4}$ by $\frac{x+6}{x+6}$ to write it as an equivalent fraction whose denominator is the LCD. We need to multiply $\frac{3}{x+6}$ by $\frac{x+4}{x+4}$ to write it as an equivalent fraction whose denominator is the LCD.

$$\frac{5}{x+4} + \frac{3}{x+6}$$

$$= \frac{5}{x+4} \cdot \frac{x+6}{x+6} + \frac{3}{x+6} \cdot \frac{x+4}{x+4}$$

Multiply to rewrite each expression as an equivalent rational expression that has the LCD as its denominator.

$$= \frac{5x+30}{(x+4)(x+6)} + \frac{3x+12}{(x+4)(x+6)}$$

Distribute in each numerator, but do not distribute in the denominators.

$$= \frac{(5x+30)+(3x+12)}{(x+4)(x+6)}$$

Add the numerators, writing the sum over the common denominator.

$$= \frac{8x+42}{(x+4)(x+6)}$$

Combine like terms.

$$= \frac{2(4x+21)}{(x+4)(x+6)}$$

Factor the numerator.

Since the numerator and denominator do not have any common factors, this rational expression cannot be simplified any further.

Quick Check 11

Add $\frac{7}{x-5} + \frac{2}{x+8}$.

A Word of Caution When adding two rational expressions, we cannot simply add the two numerators together and place their sum over the sum of the two denominators.

$$\frac{5}{x+4} + \frac{3}{x+6} \neq \frac{5+3}{(x+4)+(x+6)}$$

We must first find a common denominator and rewrite each rational expression as an equivalent expression whose denominator is equal to the common denominator.

When adding or subtracting rational expressions, leave the denominator in factored form. After we simplify the numerator, we factor the numerator if possible and check the denominator for common factors that can be divided out.

EXAMPLE 12 Add $\dfrac{x}{x^2 + 6x + 8} + \dfrac{4}{x^2 + 8x + 12}$.

Solution

In this example we must factor each denominator to find the LCD.

$$\frac{x}{x^2 + 6x + 8} + \frac{4}{x^2 + 8x + 12}$$

$$= \frac{x}{(x + 2)(x + 4)} + \frac{4}{(x + 2)(x + 6)}$$ Factor each denominator. The LCD is $(x + 2)(x + 4)(x + 6)$.

$$= \frac{x}{(x + 2)(x + 4)} \cdot \frac{x + 6}{x + 6} + \frac{4}{(x + 2)(x + 6)} \cdot \frac{x + 4}{x + 4}$$ Multiply to rewrite each expression as an equivalent rational expression that has the LCD as its denominator.

$$= \frac{x^2 + 6x}{(x + 2)(x + 4)(x + 6)} + \frac{4x + 16}{(x + 2)(x + 4)(x + 6)}$$ Distribute in each numerator.

$$= \frac{(x^2 + 6x) + (4x + 16)}{(x + 2)(x + 4)(x + 6)}$$ Add the numerators, writing the sum over the LCD.

$$= \frac{x^2 + 10x + 16}{(x + 2)(x + 4)(x + 6)}$$ Combine like terms.

$$= \frac{\overset{1}{\cancel{(x + 2)}}(x + 8)}{\underset{1}{\cancel{(x + 2)}}(x + 4)(x + 6)}$$ Factor the numerator and divide out the common factor.

$$= \frac{x + 8}{(x + 4)(x + 6)}$$ Simplify.

There is another method for creating two equivalent rational expressions that have the same denominators. In this example, once the two denominators had been factored, we had the expression $\dfrac{x}{(x + 2)(x + 4)} + \dfrac{4}{(x + 2)(x + 6)}$. Notice that the first denominator is missing the factor $x + 6$ that appears in the second denominator, so we can multiply the numerator and denominator of the first rational expression by $x + 6$. In a similar fashion, the second denominator is missing the factor $x + 4$ that appears in the first denominator, so we can multiply the numerator and denominator of the second rational expression by $x + 4$.

$$\frac{x}{(x + 2)(x + 4)} + \frac{4}{(x + 2)(x + 6)} = \frac{x}{(x + 2)(x + 4)} \cdot \frac{x + 6}{x + 6} + \frac{4}{(x + 2)(x + 6)} \cdot \frac{x + 4}{x + 4}$$

At this point, both rational expressions share the same denominator, $(x+2)(x+4)(x+6)$, so we may proceed with the addition problem.

Quick Check 12 Add $\dfrac{x}{x^2-4x+3}+\dfrac{3}{x^2+4x-5}$.

EXAMPLE 13 Subtract $\dfrac{x}{x^2-64}-\dfrac{3}{x^2-10x+16}$.

Solution

We begin by factoring each denominator to find the LCD. We must be careful that we subtract the entire second numerator, not just the first term. In other words, the subtraction must change the sign of each term in the second numerator before we combine like terms. Using parentheses around the two numerators before starting to subtract them will help with this.

$$\frac{x}{x^2-64}-\frac{3}{x^2-10x+16}$$

$$=\frac{x}{(x+8)(x-8)}-\frac{3}{(x-2)(x-8)}$$

Factor each denominator. The LCD is $(x+8)(x-8)(x-2)$.

$$=\frac{x}{(x+8)(x-8)}\cdot\frac{x-2}{x-2}-\frac{3}{(x-2)(x-8)}\cdot\frac{x+8}{x+8}$$

Multiply to rewrite each expression as an equivalent rational expression that has the LCD as its denominator.

$$=\frac{x^2-2x}{(x+8)(x-8)(x-2)}-\frac{3x+24}{(x+8)(x-8)(x-2)}$$

Distribute in each numerator.

$$=\frac{(x^2-2x)-(3x+24)}{(x+8)(x-8)(x-2)}$$

Subtract the numerators, writing the difference over the LCD.

$$=\frac{x^2-2x-3x-24}{(x+8)(x-8)(x-2)}$$

Distribute.

$$=\frac{x^2-5x-24}{(x+8)(x-8)(x-2)}$$

Combine like terms.

$$=\frac{\overset{1}{\cancel{(x-8)}}(x+3)}{(x+8)\underset{1}{\cancel{(x-8)}}(x-2)}$$

Factor the numerator and divide out the common factor.

$$=\frac{x+3}{(x+8)(x-2)}$$

Simplify.

Quick Check 13 Subtract $\dfrac{x}{x^2+x-2}-\dfrac{2}{x^2+7x+10}$.

EXAMPLE 14 Add $\dfrac{x-5}{x^2+14x+33}+\dfrac{10}{x^2+17x+66}$.

Solution

Notice that the first numerator contains a binomial. We must be careful when multiplying to create equivalent rational expressions with the same denominator. We begin by factoring each denominator to find the LCD.

$$\frac{x-5}{x^2+14x+33}+\frac{10}{x^2+17x+66}$$

$$=\frac{x-5}{(x+3)(x+11)}+\frac{10}{(x+6)(x+11)}$$

Factor each denominator. The LCD is $(x+3)(x+11)(x+6)$.

$$=\frac{x-5}{(x+3)(x+11)}\cdot\frac{x+6}{x+6}+\frac{10}{(x+6)(x+11)}\cdot\frac{x+3}{x+3}$$

Multiply to rewrite each expression as an equivalent rational expression with the LCD as its denominator.

$$=\frac{x^2+x-30}{(x+3)(x+11)(x+6)}+\frac{10x+30}{(x+3)(x+11)(x+6)}$$

Distribute in each numerator.

$$=\frac{(x^2+x-30)+(10x+30)}{(x+3)(x+11)(x+6)}$$

Add the numerators, writing the sum over the LCD.

$$=\frac{x^2+11x}{(x+3)(x+11)(x+6)}$$

Combine like terms.

$$=\frac{x\overset{1}{\cancel{(x+11)}}}{(x+3)\underset{1}{\cancel{(x-11)}}(x+6)}$$

Factor the numerator and divide out the common factor.

$$=\frac{x}{(x+3)(x+6)}$$

Simplify.

Quick Check **14** Add $\dfrac{x+6}{x^2+9x+14}+\dfrac{2}{x^2+4x-21}$.

Building Your Study Strategy **Test Taking, 3** **Write Down Important Information.** As soon as you receive your test, write down any formulas, rules, or procedures that will help you during the exam, such as a table or formula that you use to solve a particular type of word problem. Once you have written these down, you can refer to them as you work through the test. Write it down on the test while it is still fresh in your memory. That way, when you reach the appropriate problem on the test, you will not need to worry about not being able to remember the table or formula. By writing all of this information on your test, you will eliminate memorization difficulties that arise when taking a test. However, if you do not understand the material or how to use what you have written down, then you will struggle on the test. There is no substitute for understanding what you are doing.

EXERCISES 5.3

Vocabulary

1. To add fractions that have the same denominator, we add the __________ and place the result over the common denominator.

2. To subtract fractions that have the same denominator, we __________ the numerators and place the result over the common denominator.

3. When subtracting a rational expression whose numerator contains more than one term, subtract the entire numerator and not just the __________.

4. When adding two rational expressions with opposite denominators, we can replace the second denominator with its opposite by changing the addition to __________.

5. The __________ of two rational expressions is an expression that is a product of all the factors of the two denominators.

6. To add two rational expressions that have unlike denominators, we begin by converting each rational expression to a(n) __________ rational expression that has the LCD as its denominator.

Add.

7. $\dfrac{7}{x-3}+\dfrac{11}{x-3}$

8. $\dfrac{13}{x+5}+\dfrac{4}{x+5}$

9. $\dfrac{3x}{x+4}+\dfrac{12}{x+4}$

10. $\dfrac{5x}{4x+24}+\dfrac{30}{4x+24}$

11. $\dfrac{x}{x^2+3x-18}+\dfrac{6}{x^2+3x-18}$

12. $\dfrac{x}{x^2+14x+45}+\dfrac{9}{x^2+14x+45}$

13. $\dfrac{x^2+x-8}{x^2+13x+42}+\dfrac{3x-13}{x^2+13x+42}$

14. $\dfrac{x^2-7x+9}{x^2-6x-16}+\dfrac{4x-49}{x^2-6x-16}$

Subtract.

15. $\dfrac{16}{x+6}-\dfrac{7}{x+6}$

16. $\dfrac{4}{x-5}-\dfrac{14}{x-5}$

17. $\dfrac{6x}{x-7}-\dfrac{42}{x-7}$

18. $\dfrac{3x}{2x-22}-\dfrac{33}{2x-22}$

19. $\dfrac{x}{x^2-13x+36}-\dfrac{4}{x^2-13x+36}$

20. $\dfrac{x}{x^2+7x-30}-\dfrac{3}{x^2+7x-30}$

21. $\dfrac{5x-7}{x^2-36}-\dfrac{2x+11}{x^2-36}$

22. $\dfrac{12x-5}{x^2-5x-6}-\dfrac{3x-14}{x^2-5x-6}$

23. $\dfrac{x^2+8x+7}{x^2+2x-35}-\dfrac{3x+21}{x^2+2x-35}$

24. $\dfrac{x^2+4x-4}{x^2-11x+24}-\dfrac{4x+5}{x^2-11x+24}$

25. $\dfrac{2x^2+17x+23}{x^2+6x}-\dfrac{x^2+5x-13}{x^2+6x}$

26. $\dfrac{3x^2+8x-2}{x^2-25}-\dfrac{2x^2+11x+8}{x^2-25}$

Add or subtract.

27. $\dfrac{7x}{x-8}+\dfrac{56}{8-x}$

28. $\dfrac{5x}{2x-10}+\dfrac{25}{10-2x}$

FOR EXTRA HELP

MyMathLab

MathXL

Interactmath.com

MathXL Tutorials on CD

Video Lectures on CD

Addison-Wesley Math Tutor Center

Student's Solutions Manual

29. $\dfrac{x^2+4x}{x-3}-\dfrac{x-24}{3-x}$

30. $\dfrac{x^2-12x}{x-4}-\dfrac{3x+20}{4-x}$

31. $\dfrac{x^2-9x+33}{x^2-64}+\dfrac{6x-23}{64-x^2}$

32. $\dfrac{x^2-18x+10}{x^2-16}-\dfrac{3x+34}{16-x^2}$

Find the missing numerator.

33. $\dfrac{?}{x+7}+\dfrac{28}{x+7}=4$

34. $\dfrac{x^2+7x-9}{x-4}-\dfrac{?}{x-4}=x+9$

35. $\dfrac{x^2-6x-12}{(x-5)(x-2)}+\dfrac{?}{(x-5)(x-2)}=\dfrac{x+6}{x-2}$

36. $\dfrac{x^2+3x+10}{x^2+4x-32}-\dfrac{?}{x^2+4x-32}=\dfrac{x-4}{x+8}$

Find the LCD of the given rational expressions.

37. $\dfrac{9}{4x}, \dfrac{5}{6x}$

38. $\dfrac{3}{10x^3}, \dfrac{7}{8x^5}$

39. $\dfrac{11}{x-8}, \dfrac{1}{x+4}$

40. $\dfrac{x-3}{x-4}, \dfrac{x+6}{2x+5}$

41. $\dfrac{x+4}{x^2+9x+14}, \dfrac{x}{x^2-4}$

42. $\dfrac{x-5}{x^2+3x-54}, \dfrac{x+4}{x^2-x-30}$

43. $\dfrac{3x-8}{x^2-6x+9}, \dfrac{x+1}{x^2-7x+12}$

44. $\dfrac{x-6}{x^2+14x+49}, \dfrac{9}{x^2+7x}$

Add or subtract.

45. $\dfrac{5x}{6}+\dfrac{2x}{9}$

46. $\dfrac{7x}{12}-\dfrac{x}{20}$

47. $\dfrac{9}{8x}-\dfrac{11}{6x}$

48. $\dfrac{7}{9x}+\dfrac{3}{10x}$

49. $\dfrac{2}{x+5}+\dfrac{3}{x+4}$

50. $\dfrac{3}{x-6}-\dfrac{7}{x+1}$

51. $\dfrac{5}{(x-10)(x-5)}-\dfrac{4}{(x-5)(x-9)}$

52. $\dfrac{8}{(x+2)(x+6)}+\dfrac{2}{(x+6)(x+7)}$

53. $\dfrac{9}{x^2+x-2}+\dfrac{6}{x^2-4x+3}$

54. $\dfrac{3}{x^2+7x+10}+\dfrac{5}{x^2-x-6}$

55. $\dfrac{5}{x^2-16x+63}-\dfrac{2}{x^2-15x+56}$

56. $\dfrac{7}{x^2+8x+15}-\dfrac{4}{x^2+7x+12}$

57. $\dfrac{1}{x^2-15x+54}-\dfrac{6}{x^2-81}$

58. $\dfrac{4}{x^2-4}-\dfrac{7}{x^2-3x-10}$

59. $\dfrac{5}{2x^2+3x-2}+\dfrac{7}{2x^2-9x+4}$

60. $\dfrac{7}{2x^2-15x-27}-\dfrac{3}{2x^2-3x-9}$

61. $\dfrac{x+3}{x^2+11x+30}+\dfrac{4}{x^2+12x+35}$

62. $\dfrac{x-1}{x^2+13x+40}+\dfrac{3}{x^2+15x+56}$

63. $\dfrac{x+2}{x^2-36}-\dfrac{2}{x^2-9x+18}$

64. $\dfrac{x-4}{x^2-8x+15}-\dfrac{6}{x^2+2x-35}$

65. $\dfrac{x+2}{x^2+6x-16}+\dfrac{3}{x^2+11x+24}$

66. $\dfrac{x+5}{x^2-2x-3}+\dfrac{1}{x^2+3x+2}$

67. $\dfrac{x-1}{x^2+7x+10}-\dfrac{3}{x^2+x-2}$

68. $\dfrac{x+7}{x^2+5x+4}-\dfrac{4}{x^2+4x+3}$

69. $\dfrac{x+10}{x^2+15x+54}+\dfrac{7}{x^2+6x}$

70. $\dfrac{x+3}{x^2-7x}-\dfrac{5}{x^2-49}$

71. $\dfrac{x+1}{x^2-5x+6}+\dfrac{x-8}{x^2-6x+8}$

72. $\dfrac{x+1}{x^2+9x+20}+\dfrac{x+10}{x^2+10x+24}$

For the given rational functions f(x) and g(x), find f(x) + g(x).

73. $f(x)=\dfrac{3x}{4x+28},\ g(x)=\dfrac{21}{4x+28}$

74. $f(x)=\dfrac{x^2+7x+11}{x^2-9},\ g(x)=\dfrac{2x+5}{9-x^2}$

75. $f(x)=\dfrac{3}{x^2-3x-18},\ g(x)=\dfrac{2}{x^2+12x+27}$

76. $f(x)=\dfrac{x+5}{x^2+x},\ g(x)=\dfrac{5}{x^2-x}$

For the given rational functions f(x) and g(x), find f(x) − g(x).

77. $f(x)=\dfrac{13}{x-3},\ g(x)=\dfrac{4}{x-3}$

78. $f(x)=\dfrac{x^2+4x}{x-6},\ g(x)=\dfrac{x+6}{6-x}$

79. $f(x)=\dfrac{6}{x^2+12x+27},\ g(x)=\dfrac{2}{x^2+16x+63}$

80. $f(x)=\dfrac{x+6}{x^2+18x+80},\ g(x)=\dfrac{2}{x^2+14x+48}$

Mixed Practice, 81–98

Add or subtract.

81. $\dfrac{x-2}{15}+\dfrac{x+3}{18}$

82. $\dfrac{x-6}{x^2+9x-36}+\dfrac{2}{x^2+4x-21}$

83. $\dfrac{4}{x^2-10x+21}+\dfrac{5}{x^2-x-42}$

84. $\dfrac{3x+7}{x-7}-\dfrac{x-35}{7-x}$

85. $\dfrac{x^2-10x+2}{x^2+9x-10}+\dfrac{3x+4}{x^2+9x-10}$

86. $\dfrac{8}{x^2-x-20}-\dfrac{6}{x^2-16}$

87. $\dfrac{x^2+5x+21}{x^2-81}-\dfrac{10x+33}{81-x^2}$

88. $\dfrac{11}{6a}-\dfrac{7}{10a}$

89. $\dfrac{3x-8}{x^2-25}-\dfrac{x+2}{x^2-25}$

90. $\dfrac{x^2-2x+26}{x^2+x-56}-\dfrac{8x+5}{x^2+x-56}$

91. $\dfrac{x+5}{x^2-17x+70}-\dfrac{4}{x^2-15x+56}$

92. $\dfrac{x+8}{x^2-16} - \dfrac{2}{x^2+4x}$

93. $\dfrac{5}{3x-4} + \dfrac{2}{4-3x}$

94. $\dfrac{3}{x^2+9x+18} + \dfrac{7}{x^2-x-12}$

95. $\dfrac{x-5}{x^2+5x+4} + \dfrac{6}{x^2+6x+8}$

96. $\dfrac{x^2-6x+24}{x^2-8x} + \dfrac{7x-16}{8x-x^2}$

97. $\dfrac{2}{x^2+6x+8} - \dfrac{3}{x^2+7x+10}$

98. $\dfrac{x^2-7x-9}{x^2+8x+12} + \dfrac{11x+13}{x^2+8x+12}$

Writing in Mathematics

Answer in complete sentences.

99. Explain how to determine that two rational expressions have opposite denominators.

100. Here is a student's solution to a problem on an exam. Describe the student's error, and provide the correct solution. Assuming that the problem was worth 10 points, how many points would you give to the student for his solution? Explain your reasoning.

$$\frac{5}{(x+3)(x+2)} + \frac{7}{(x+3)(x+6)}$$
$$= \frac{5}{(x+3)(x+2)} \cdot \frac{x+6}{x+6} + \frac{7}{(x+3)(x+6)} \cdot \frac{x+2}{x+2}$$
$$= \frac{5(x+6)+7(x+2)}{(x+3)(x+2)(x+6)}$$
$$= \frac{5\overset{1}{\cancel{(x+6)}}+7\overset{1}{\cancel{(x+2)}}}{(x+3)\underset{1}{\cancel{(x+2)}}\underset{1}{\cancel{(x+6)}}}$$
$$= \frac{12}{x+3}$$

101. ***Solutions Manual**** Write a solutions manual page for the following problem:

Subtract $\dfrac{x^2+3x+3}{x^2+6x-16} - \dfrac{12x-11}{x^2+6x-16}$.

102. ***Solutions Manual**** Write a solutions manual page for the following problem:

Add $\dfrac{x+6}{x^2+9x+20} + \dfrac{4}{x^2+6x+8}$.

103. ***Newsletter**** Write a newsletter that explains how to add two rational expressions with the same denominator.

104. ***Newsletter**** Write a newsletter that explains how to subtract two rational expressions with unlike denominators.

***See Appendix B for details and sample answers.**

5.4 Complex Fractions

Objectives

1. Simplify complex numerical fractions.
2. Simplify complex fractions containing variables.

Complex Fractions

A **complex fraction** is a fraction or rational expression containing one or more fractions in its numerator or denominator. Here are some examples.

$$\frac{\frac{1}{2}+\frac{5}{3}}{\frac{10}{3}-\frac{7}{4}} \qquad \frac{\frac{x+5}{x-9}}{1-\frac{5}{x}} \qquad \frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} \qquad \frac{1+\frac{3}{x}-\frac{28}{x^2}}{1+\frac{7}{x}}$$

Objective 1 **Simplify complex numerical fractions.** To simplify a complex fraction, we must rewrite it in such a way that its numerator and denominator do not contain fractions. This can be done by finding the LCD of all fractions within the complex fraction and then multiplying the numerator and denominator by this LCD. This will clear the fractions within the complex fraction. We finish by simplifying the resulting rational expression, if possible.

We begin with a complex fraction made up of numerical fractions.

Example 1 Simplify the complex fraction $\dfrac{3+\frac{5}{6}}{\frac{2}{3}+\frac{3}{4}}$.

Solution

The LCD of the three denominators (6, 3, and 4) is 12, so we begin by multiplying the complex fraction by $\frac{12}{12}$. Notice that when we multiply by $\frac{12}{12}$, we are really multiplying by 1, which does not change the value of the original expression.

$$\frac{3+\frac{5}{6}}{\frac{2}{3}+\frac{3}{4}} = \frac{12}{12}\cdot\frac{3+\frac{5}{6}}{\frac{2}{3}+\frac{3}{4}}$$

Multiply the numerator and denominator by the LCD.

$$= \frac{12\cdot 3+\overset{2}{\cancel{12}}\cdot\frac{5}{\underset{1}{\cancel{6}}}}{\overset{4}{\cancel{12}}\cdot\frac{2}{\underset{1}{\cancel{3}}}+\overset{3}{\cancel{12}}\cdot\frac{3}{\underset{1}{\cancel{4}}}}$$

Distribute and divide out common factors.

$$= \frac{36 + 10}{8 + 9} \quad \text{Multiply.}$$

$$= \frac{46}{17} \quad \text{Simplify the numerator and denominator.}$$

There is another method for simplifying complex fractions. We can rewrite the numerator as a single fraction by adding $3 + \frac{5}{6}$, which would equal $\frac{23}{6}$.

$$3 + \frac{5}{6} = \frac{18}{6} + \frac{5}{6} = \frac{23}{6}$$

We can also rewrite the denominator as a single fraction by adding $\frac{2}{3} + \frac{3}{4}$, which would equal $\frac{17}{12}$.

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$$

Once the numerator and denominator are single fractions, we can rewrite $\dfrac{\frac{23}{6}}{\frac{17}{12}}$ as a division problem $\left(\frac{23}{6} \div \frac{17}{12}\right)$ and simplify from there. This method produces the same result, $\frac{46}{17}$.

$$\frac{23}{6} \div \frac{17}{12} = \frac{23}{\cancelto{1}{6}} \cdot \frac{\cancelto{2}{12}}{17} = \frac{46}{17}$$

In most of the remaining examples we will use the LCD method, as the technique is similar to the technique used to solve rational equations in the next section.

Quick Check 1

Simplify the complex fraction $\dfrac{\frac{5}{4} + \frac{8}{5}}{\frac{3}{10} - \frac{1}{8}}$.

Simplifying Complex Fractions Containing Variables

Objective 2 **Simplify complex fractions containing variables.**

EXAMPLE 2 Simplify $\dfrac{1 + \frac{5}{x}}{1 - \frac{25}{x^2}}$.

Solution

The LCD for the two simple fractions with denominators x and x^2 is x^2, so we will begin by multiplying the complex fraction by $\frac{x^2}{x^2}$.

$$\frac{1 + \frac{5}{x}}{1 - \frac{25}{x^2}} = \frac{x^2}{x^2} \cdot \frac{1 + \frac{5}{x}}{1 - \frac{25}{x^2}} \quad \text{Multiply the numerator and denominator by the LCD.}$$

$$= \frac{x^2 \cdot 1 + \cancel{x^2}^{x} \cdot \frac{5}{\cancel{x}_1}}{x^2 \cdot 1 - \cancel{x^2}^{1} \cdot \frac{25}{\cancel{x^2}_1}}$$ Distribute and divide out common factors, clearing the fractions.

$$= \frac{x^2 + 5x}{x^2 - 25}$$ Multiply.

$$= \frac{x\cancel{(x+5)}^{1}}{\cancel{(x+5)}_{1}(x-5)}$$ Factor the numerator and denominator and divide out the common factor.

$$= \frac{x}{x-5}$$ Simplify.

Once we have cleared the fractions, resulting in a rational expression such as $\frac{x^2 + 5x}{x^2 - 25}$, then we simplify the rational expression by factoring and dividing out common factors.

Quick Check 2

Simplify $\dfrac{\frac{1}{49} - \frac{1}{x^2}}{\frac{1}{7} + \frac{1}{x}}$.

EXAMPLE 3 Simplify $\dfrac{1 + \frac{6}{x} - \frac{16}{x^2}}{1 - \frac{5}{x} + \frac{6}{x^2}}$.

Solution

We begin by multiplying the numerator and denominator by the LCD of all denominators. In this case the LCD is x^2.

$$\frac{1 + \frac{6}{x} - \frac{16}{x^2}}{1 - \frac{5}{x} + \frac{6}{x^2}} = \frac{x^2}{x^2} \cdot \frac{1 + \frac{6}{x} - \frac{16}{x^2}}{1 - \frac{5}{x} + \frac{6}{x^2}}$$ Multiply the numerator and denominator by the LCD.

$$= \frac{x^2 \cdot 1 + \cancel{x^2}^{x} \frac{6}{\cancel{x}_1} - \cancel{x^2}^{1} \frac{16}{\cancel{x^2}_1}}{x^2 \cdot 1 - \cancel{x^2}^{x} \cdot \frac{5}{\cancel{x}_1} + \cancel{x^2}^{1} \cdot \frac{6}{\cancel{x^2}_1}}$$ Distribute and divide out common factors.

$$= \frac{x^2 + 6x - 16}{x^2 - 5x + 6}$$ Multiply.

$$= \frac{(x+8)\cancel{(x-2)}^{1}}{\cancel{(x-2)}_{1}(x-3)}$$ Factor the numerator and denominator and divide out the common factor.

$$= \frac{x+8}{x-3}$$ Simplify.

Quick Check 3

Simplify $\dfrac{1 - \frac{8}{x} - \frac{9}{x^2}}{1 - \frac{81}{x^2}}$.

EXAMPLE 4 Simplify $\dfrac{\dfrac{6}{x+5}-\dfrac{3}{x+3}}{\dfrac{x}{x+3}+\dfrac{2}{x+5}}$.

Solution

The LCD for the four simple fractions is $(x+3)(x+5)$.

$$\dfrac{\dfrac{6}{x+5}-\dfrac{3}{x+3}}{\dfrac{x}{x+3}+\dfrac{2}{x+5}}$$

$$=\dfrac{(x+3)(x+5)}{(x+3)(x+5)}\cdot\dfrac{\dfrac{6}{x+5}-\dfrac{3}{x+3}}{\dfrac{x}{x+3}+\dfrac{2}{x+5}}$$ Multiply the numerator and denominator by the LCD

$$=\dfrac{(x+3)\overset{1}{\cancel{(x+5)}}\cdot\dfrac{6}{\underset{1}{\cancel{x+5}}}-\overset{1}{\cancel{(x+3)}}(x+5)\cdot\dfrac{3}{\underset{1}{\cancel{x+3}}}}{\overset{1}{\cancel{(x+3)}}(x+5)\cdot\dfrac{x}{\underset{1}{\cancel{x+3}}}+(x+3)\overset{1}{\cancel{(x+5)}}\dfrac{2}{\underset{1}{\cancel{x+5}}}}$$ Distribute and divide out common factors, clearing the fractions.

$$=\dfrac{6(x+3)-3(x+5)}{x(x+5)+2(x+3)}$$ Multiply.

$$=\dfrac{6x+18-3x-15}{x^2+5x+2x+6}$$ Distribute.

$$=\dfrac{3x+3}{x^2+7x+6}$$ Combine like terms.

$$=\dfrac{3\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+1)}}(x+6)}$$ Factor the denominator and divide out the common factor.

$$=\dfrac{3}{x+6}$$ Simplify.

Quick Check 4

Simplify $\dfrac{\dfrac{1}{x+6}+\dfrac{1}{x-2}}{\dfrac{x}{x+6}-\dfrac{2}{x-2}}$.

EXAMPLE 5 Simplify $\dfrac{\dfrac{x^2+10x+21}{x^2-5x-36}}{\dfrac{x^2-9}{x^2+12x+32}}$.

Solution

In this case it will be easier to rewrite the complex fraction as a division problem rather than multiplying the numerator and denominator by the LCD. This is a wise idea when we have a complex fraction with a single rational expression in its numerator and a single rational expression in its denominator. We have used this method when dividing rational expressions.

$$\frac{\dfrac{x^2+10x+21}{x^2-5x-36}}{\dfrac{x^2-9}{x^2+12x+32}} = \frac{x^2+10x+21}{x^2-5x-36} \div \frac{x^2-9}{x^2+12x+32}$$ Rewrite as a division problem.

$$= \frac{x^2+10x+21}{x^2-5x-36} \cdot \frac{x^2+12x+32}{x^2-9}$$ Invert the divisor and multiply.

$$= \frac{(x+3)(x+7)}{(x+4)(x-9)} \cdot \frac{(x+4)(x+8)}{(x+3)(x-3)}$$ Factor each numerator and denominator.

$$= \frac{\overset{1}{\cancel{(x+3)}}(x+7)}{\underset{1}{\cancel{(x+4)}}(x-9)} \cdot \frac{\overset{1}{\cancel{(x+4)}}(x+8)}{\underset{1}{\cancel{(x+3)}}(x-3)}$$ Divide out common factors.

$$= \frac{(x+7)(x+8)}{(x-9)(x-3)}$$ Simplify.

Quick Check 5

Simplify $\dfrac{\dfrac{x^2-8x+16}{x^2+8x-9}}{\dfrac{x^2+x-20}{x^2+2x-63}}$.

Building Your Study Strategy **Test Taking, 4** **Read the Test** In the same way you would begin to solve a word problem, you should begin to take a test by briefly reading through it. This will give you an idea of how many problems you have to solve, how many word problems there are, and roughly how much time you can devote to each problem. It is a good idea to establish a schedule, such as "I need to be done with 12 problems by the time half of the class period is over." This way you will know whether you need to speed up during the second half of the exam or if you have plenty of time.

EXERCISES 5.4

Vocabulary

1. A(n) __________ is a fraction or rational expression containing one or more fractions in its numerator or denominator.

2. To simplify a complex fraction, multiply the numerator and denominator by the __________ of all fractions within the complex fraction.

Simplify the complex fraction.

3. $\dfrac{\frac{3}{5}-\frac{2}{3}}{\frac{1}{6}+\frac{3}{10}}$

4. $\dfrac{\frac{5}{6}+\frac{11}{8}}{\frac{3}{4}-\frac{1}{12}}$

5. $\dfrac{3-\frac{3}{4}}{\frac{7}{8}-\frac{1}{6}}$

6. $\dfrac{\frac{7}{10}-3}{\frac{3}{5}+\frac{5}{4}}$

FOR EXTRA HELP

MyMathLab

MathXL

Interactmath.com

MathXL Tutorials on CD

Video Lectures on CD

Tutor Center

Addison-Wesley Math Tutor Center

Student's Solutions Manual

7. $\dfrac{x+\dfrac{2}{9}}{x+\dfrac{3}{2}}$

8. $\dfrac{x-\dfrac{4}{5}}{x+\dfrac{3}{8}}$

9. $\dfrac{10-\dfrac{15}{x}}{x-\dfrac{3}{2}}$

10. $\dfrac{3+\dfrac{4}{x}}{\dfrac{x}{2}+\dfrac{2}{3}}$

11. $\dfrac{4+\dfrac{12}{x}}{1-\dfrac{9}{x^2}}$

12. $\dfrac{3-\dfrac{108}{x^2}}{1+\dfrac{6}{x}}$

13. $\dfrac{\dfrac{1}{9}-\dfrac{1}{x^2}}{\dfrac{1}{3}-\dfrac{1}{x}}$

14. $\dfrac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{x^2}-\dfrac{1}{25}}$

15. $\dfrac{\dfrac{7}{x-5}-\dfrac{4}{x-2}}{\dfrac{x+2}{x-5}}$

16. $\dfrac{\dfrac{5}{x-9}+\dfrac{3}{x-1}}{2-\dfrac{x+2}{x-1}}$

17. $\dfrac{\dfrac{6}{x+7}+\dfrac{5}{x-4}}{2-\dfrac{x+13}{x+7}}$

18. $\dfrac{\dfrac{4}{x+4}-\dfrac{5}{x+3}}{5+\dfrac{x+33}{x+3}}$

19. $\dfrac{1+\dfrac{3}{x}-\dfrac{10}{x^2}}{1-\dfrac{1}{x}-\dfrac{30}{x^2}}$

20. $\dfrac{1+\dfrac{8}{x}+\dfrac{16}{x^2}}{1+\dfrac{13}{x}+\dfrac{36}{x^2}}$

21. $\dfrac{1-\dfrac{5}{x}-\dfrac{24}{x^2}}{1-\dfrac{64}{x^2}}$

22. $\dfrac{1-\dfrac{4}{x^2}}{1+\dfrac{1}{x}-\dfrac{6}{x^2}}$

23. $\dfrac{x+7+\dfrac{6}{x}}{1-\dfrac{4}{x}-\dfrac{5}{x^2}}$

24. $\dfrac{1-\dfrac{14}{x}+\dfrac{40}{x^2}}{x-7+\dfrac{12}{x}}$

25. $\dfrac{\dfrac{x^2+12x+35}{x^2-3x-40}}{\dfrac{x^2+9x+14}{x^2-64}}$

26. $\dfrac{\dfrac{x^2-7x+6}{x^2-16}}{\dfrac{x^2-6x}{x^2+13x+36}}$

27. $\dfrac{\dfrac{x^2+17x+72}{x^2+5x+6}}{\dfrac{x^2-81}{x^2-2x-15}}$

28. $\dfrac{\dfrac{x^2-2x}{x^2-10x+16}}{\dfrac{x^2+4x-60}{x^2-14x+48}}$

29. $\dfrac{\dfrac{1}{x^2+15x+36}}{\dfrac{1}{x^2-7x-30}}$

30. $\dfrac{\dfrac{1}{x^2-25x+100}}{\dfrac{1}{x^2+15x-100}}$

Mixed Practice, 31–54

Simplify the given rational expression, using the techniques developed in Sections 5.1 through 5.4.

31. $\dfrac{x^2+3x+5}{x^2+2x-48}+\dfrac{2x-29}{x^2+2x-48}$

32. $\dfrac{x^2+5x+6}{x^2+5x-6}\div\dfrac{x^2-9}{x^2-x-42}$

33. $\dfrac{x+3}{x^2-2x-24}+\dfrac{2}{x^2-8x+12}$

34. $\dfrac{5}{x^2+4x-96}-\dfrac{1}{x^2-12x+32}$

35. $\dfrac{x^2+3x}{x^2-9x+18}\div\dfrac{x^2-25}{x^2-11x+30}$

36. $\dfrac{x^2-64}{24x-3x^2}$

37. $\dfrac{x-1}{x^2-3x} - \dfrac{6}{x^2+3x-18}$

38. $\dfrac{x^2+12x+20}{x^2+11x+30} \cdot \dfrac{x^2+x-30}{x^2+13x+30}$

39. $\dfrac{9x}{2x-6} + \dfrac{27}{6-2x}$

40. $\dfrac{1-\dfrac{18}{x}+\dfrac{81}{x^2}}{1-\dfrac{1}{x}-\dfrac{72}{x^2}}$

41. $\dfrac{\dfrac{3}{x-3}+\dfrac{2}{x+12}}{\dfrac{x+6}{x-3}}$

42. $\dfrac{3}{x^2+7x+12} + \dfrac{9}{x^2+11x+28}$

43. $\dfrac{9}{x^2+13x+22} - \dfrac{4}{x^2+20x+99}$

44. $\dfrac{4x-x^2}{x^2+13x+42} \cdot \dfrac{49-x^2}{x^2-x-12}$

45. $\dfrac{\dfrac{1}{3}-\dfrac{1}{x}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

46. $\dfrac{x^2+5x+20}{x^2-4} - \dfrac{5x-4}{4-x^2}$

47. $\dfrac{5}{x^2+7x+12} + \dfrac{6}{x^2+10x+24}$

48. $\dfrac{6x}{2x-9} + \dfrac{27}{9-2x}$

49. $\dfrac{x^2+13x+30}{x^2+13x-30} \cdot \dfrac{x^2-5x+6}{x^2-9}$

50. $\dfrac{x+6}{x^2+9x+20} + \dfrac{4}{x^2+6x+8}$

51. $\dfrac{x^2+4x-10}{x^2-5x-14} - \dfrac{5x+32}{x^2-5x-14}$

52. $\dfrac{x+\dfrac{3}{10}}{x-\dfrac{2}{5}}$

53. $\dfrac{x^2+8x}{x^2+15x-16} \div \dfrac{x^2-64}{x^2-10x+9}$

54. $\dfrac{x-5}{x^2-4x+3} - \dfrac{6}{x^2+x-2}$

Writing in Mathematics

Answer in complete sentences.

55. Explain what a complex fraction is. Compare and contrast complex fractions and the rational expressions found in Section 5.1.

56. One method for simplifying complex fractions is to rewrite the complex fraction as one rational expression divided by another rational expression. When is this method the most efficient way to simplify a complex fraction?

57. ***Solutions Manual**** Write a solutions manual page for the following problem:

Simplify $\dfrac{\dfrac{3}{x-9}+\dfrac{6}{x-3}}{3-\dfrac{x+5}{x-3}}$.

58. ***Newsletter**** Write a newsletter that explains how to simplify complex fractions.

***See Appendix B for details and sample answers.**

5.5 Rational Equations

Objectives

1. **Solve rational equations.**
2. **Solve literal equations.**

Solving Rational Equations

Objective 1 Solve rational equations. In this section, we learn how to solve **rational equations**, which are equations containing at least one rational expression. The main goal is to rewrite the equation as an equivalent equation that does not contain a rational expression. We then solve the equation using methods developed in earlier chapters.

In Chapter 1, we learned how to solve an equation containing fractions such as the equation $\frac{1}{4}x - \frac{3}{5} = \frac{9}{10}$. We began by finding the LCD of all fractions, and then multiplied both sides of the equation by that LCD to clear the equation of fractions. We will employ the same technique in this section. There is a major difference, though, when solving equations containing a variable in a denominator. Occasionally we will find a solution that causes one of the rational expressions in the equation to be undefined. If a denominator of a rational expression is equal to 0 when the value of a solution is substituted for the variable, then that solution must be omitted from the answer set and is called an **extraneous solution**. We must check each solution that we find to make sure that it is not an extraneous solution.

Here is a summary of the method for solving rational equations:

Solving Rational Equations

1. Find the LCD of all denominators in the equation.
2. Multiply both sides of the equation by the LCD to clear the equation of fractions.
3. Solve the resulting equation.
4. Check for extraneous solutions.

Example 1 Solve $\frac{9}{x} + \frac{2}{3} = \frac{17}{12}$.

Solution

We begin by finding the LCD of these three fractions, which is $12x$. Now we multiply both sides of the equation by the LCD to clear the equation of fractions. Once this has been done, we can solve the resulting equation.

$$\frac{9}{x} + \frac{2}{3} = \frac{17}{12}$$

$$12x\left(\frac{9}{x} + \frac{2}{3}\right) = 12x \cdot \frac{17}{12} \qquad \text{Multiply both sides of the equation by the LCD.}$$

$$12\overset{1}{\cancel{x}} \cdot \frac{9}{\underset{1}{\cancel{x}}} + \overset{4}{\cancel{12}}x \cdot \frac{2}{\underset{1}{\cancel{3}}} = \overset{1}{\cancel{12}}x \cdot \frac{17}{\underset{1}{\cancel{12}}} \qquad \text{Distribute and divide out common factors.}$$

$$108 + 8x = 17x \qquad \text{Multiply. The resulting equation is linear.}$$

$108 = 9x$ Subtract $8x$ to collect all variable terms on one side of the equation.

$12 = x$ Divide both sides by 9.

Check:

$$\frac{9}{(12)} + \frac{2}{3} = \frac{17}{12}$$ Substitute 12 for x.

$$\frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$ Simplify the fraction $\frac{9}{12}$. The LCD of these fractions is 12.

$$\frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$ Write each fraction with a common denominator of 12.

$$\frac{17}{12} = \frac{17}{12}$$ Add.

Since $x = 12$ does not make any rational expression in the original equation undefined, this value is a solution. The solution set is $\{12\}$.

***Quick Check* 1**

Solve $\frac{38}{x} - \frac{3}{5} = \frac{2}{3}$.

When checking whether a solution is an extraneous solution, we need only determine whether the solution causes the LCD to equal 0. If the LCD is equal to 0 for this solution, then one or more rational expressions are undefined and the solution is an extraneous solution. Also, if the LCD is equal to 0, then we have multiplied both sides of the equation by 0. The multiplication property of equality says we can multiply both sides of an equation by any *nonzero* number without affecting the equality of both sides. In the previous example, the only solution that could possibly be an extraneous solution is $x = 0$, because that is the only value of x for which the LCD is equal to 0.

EXAMPLE 2 Solve $x + 6 - \frac{27}{x} = 0$.

Solution

The LCD in this example is x. The LCD is equal to 0 only if $x = 0$. If we find that $x = 0$ is a solution, then we must omit that solution as an extraneous solution.

$$x + 6 - \frac{27}{x} = 0$$

$$x \cdot \left(x + 6 - \frac{27}{x}\right) = x \cdot 0$$ Multiply each side of the equation by the LCD, x.

$$x \cdot x + x \cdot 6 - \overset{1}{\cancel{x}} \cdot \frac{27}{\underset{1}{\cancel{x}}} = x \cdot 0$$ Distribute and divide out common factors.

$$x^2 + 6x - 27 = 0$$ Multiply. The resulting equation is quadratic.

$$(x + 9)(x - 3) = 0$$ Factor.

$$x = -9 \quad \text{or} \quad x = 3$$ Set each factor equal to 0 and solve.

You may verify that neither solution causes the LCD to equal 0. The solution set is $\{-9, 3\}$.

***Quick Check* 2**

Solve $1 = \frac{4}{x} + \frac{32}{x^2}$.

A Word of Caution When solving a rational equation, the use of the LCD is completely different than when we are adding or subtracting rational expressions. We use the LCD to clear the denominators of the rational expressions when solving a rational equation. When adding or subtracting rational expressions, we rewrite each expression as an equivalent expression whose denominator is the LCD.

EXAMPLE 3 Solve $\frac{x+5}{x-4} + 3 = \frac{2x+1}{x-4}$.

Solution

The LCD is $x - 4$, so we will begin to solve this equation by multiplying both sides of the equation by $x - 4$.

$$\frac{x+5}{x-4} + 3 = \frac{2x+1}{x-4}$$ The LCD is $x - 4$.

$$(x-4)\left(\frac{x+5}{x-4} + 3\right) = (x-4)\cdot\frac{2x+1}{x-4}$$ Multiply both sides by the LCD.

$$\cancel{(x-4)}^{1}\frac{x+5}{\cancel{x-4}_{1}} + (x-4)3 = \cancel{(x-4)}^{1}\cdot\frac{2x+1}{\cancel{x-4}_{1}}$$ Distribute and divide out common factors.

$$x + 5 + 3x - 12 = 2x + 1$$ Multiply. The resulting equation is linear.

$$4x - 7 = 2x + 1$$ Combine like terms.

$$2x - 7 = 1$$ Subtract $2x$.

$$2x = 8$$ Add 7.

$$x = 4$$ Divide both sides by 2.

The LCD is equal to 0 when $x = 4$, and two rational expressions in the original equation are undefined when $x = 4$.

Check:

$$\frac{(4)+5}{(4)-4} + 3 = \frac{2(4)+1}{(4)-4}$$ Substitute 4 for x.

$$\frac{9}{0} + 3 = \frac{9}{0}$$ Simplify each numerator and denominator.

***Quick Check* 3**

Solve

$$\frac{x+3}{x+2} - 4 = \frac{3x+7}{x+2}.$$

So, although $x = 4$ is a solution to the equation obtained once the fractions have been cleared, it is not a solution to the original equation. This solution is an extraneous solution and since there are no other solutions, this equation has no solution. Recall that we write the solution set as $\emptyset$ when there is no solution.

EXAMPLE 4 Solve $\frac{x-1}{x^2+4x+3} = \frac{5}{x^2-3x-4}$.

Solution

We begin by factoring the denominators to find the LCD.

$$\frac{x-1}{x^2+4x+3} = \frac{5}{x^2-3x-4}$$

$$\frac{x-1}{(x+1)(x+3)} = \frac{5}{(x-4)(x+1)}$$ The LCD is $(x+1)(x+3)(x-4)$.

$$\cancel{(x+1)}\cancel{(x+3)}(x-4)\cdot\frac{x-1}{\cancel{(x+1)}\cancel{(x+3)}} = \cancel{(x+1)}(x+3)\cancel{(x-4)}\cdot\frac{5}{\cancel{(x-4)}\cancel{(x+1)}}$$ Multiply by the LCD. Divide out common factors.

$$(x-4)(x-1) = 5(x+3)$$ Multiply remaining factors.

$$x^2 - 5x + 4 = 5x + 15$$ Multiply.

$$x^2 - 10x - 11 = 0$$ Collect all terms on the left side. The resulting equation is quadratic.

$$(x+1)(x-11) = 0$$ Factor.

$$x = -1 \quad \text{or} \quad x = 11$$ Set each factor equal to 0 and solve.

The solution $x = -1$ is an extraneous solution as it makes the LCD equal to 0. It is left to the reader to verify that the solution $x = 11$ checks. The solution set for this equation is $\{11\}$.

Quick Check **4** **Solve** $\dfrac{x+8}{x^2-10x+24} = \dfrac{6}{x^2-9x+20}$.

EXAMPLE 5 Solve $\dfrac{x}{x+5} + \dfrac{3}{x+8} = \dfrac{7x+20}{x^2+13x+40}$.

Solution

$$\frac{x}{x+5} + \frac{3}{x+8} = \frac{7x+20}{x^2+13x+40}$$

$$\frac{x}{x+5} + \frac{3}{x+8} = \frac{7x+20}{(x+5)(x+8)}$$ The LCD is $(x+5)(x+8)$.

$$(x+5)(x+8)\left(\frac{x}{x+5} + \frac{3}{x+8}\right) = (x+5)(x+8)\cdot\frac{7x+20}{(x+5)(x+8)}$$ Multiply by the LCD.

$$\cancel{(x+5)}(x+8)\cdot\frac{x}{\cancel{(x+5)}} + (x+5)\cancel{(x+8)}\cdot\frac{3}{\cancel{(x+8)}} = \cancel{(x+5)}\cancel{(x+8)}\cdot\frac{7x+20}{\cancel{(x+5)}\cancel{(x+8)}}$$ Distribute and divide out common factors.

$$x(x+8) + 3(x+5) = 7x + 20$$ Multiply remaining factors.

$$x^2 + 8x + 3x + 15 = 7x + 20$$ Distribute.

$$x^2 + 11x + 15 = 7x + 20$$ Combine like terms.

$$x^2 + 4x - 5 = 0$$ Collect all terms on the left side. The resulting equation is quadratic.

$$(x + 5)(x - 1) = 0$$ Factor.

$$x = -5 \quad \text{or} \quad x = 1$$ Set each factor equal to 0 and solve.

The solution $x = -5$ is an extraneous solution as it makes the LCD equal to 0. It is left to the reader to verify that the solution $x = 1$ checks. The solution set for this equation is $\{1\}$.

Quick Check **5** Solve $\dfrac{x}{x-3} - \dfrac{6}{x-9} = \dfrac{x-45}{x^2-12x+27}$.

Literal Equations

Objective 2 **Solve literal equations.** Recall that a literal equation is an equation containing two or more variables, and we solve the equation for one of the variables by isolating that variable on one side of the equation. In this section, we will learn how to solve literal equations containing one or more rational expressions.

EXAMPLE 6 Solve the literal equation $\dfrac{2}{x} + \dfrac{5}{y} = \dfrac{3}{4}$ for x.

Solution

We begin by multiplying by the LCD $(4xy)$ to clear the equation of fractions.

$$\frac{2}{x} + \frac{5}{y} = \frac{3}{4}$$

$$4xy\left(\frac{2}{x} + \frac{5}{y}\right) = 4xy \cdot \frac{3}{4}$$ Multiply by the LCD, $4xy$.

$$4\overset{1}{\cancel{x}}y \cdot \frac{2}{\underset{1}{\cancel{x}}} + 4x\overset{1}{\cancel{y}} \cdot \frac{5}{\underset{1}{\cancel{y}}} = \overset{1}{\cancel{4}}xy \cdot \frac{3}{\underset{1}{\cancel{4}}}$$ Distribute and divide out common factors.

$$8y + 20x = 3xy$$ Multiply remaining factors.

Notice that there are two terms that contain the variable we are solving for. We need to collect both of these terms on the same side of the equation and then factor x out of those terms. This will allow us to divide and isolate x.

$$8y + 20x = 3xy$$

$$8y = 3xy - 20x$$ Subtract $20x$ to collect all terms with x on the right side of the equation.

$$8y = x(3y - 20)$$ Factor out the common factor x.

$$\frac{8y}{3y - 20} = \frac{x\overset{1}{\cancel{(3y - 20)}}}{\underset{1}{\cancel{3y - 20}}}$$ Divide both sides by $3y - 20$ to isolate x.

$$\frac{8y}{3y-20} = x$$ Simplify.

$$x = \frac{8y}{3y-20}$$ Rewrite with the variable we are solving for on the left side of the equation.

Quick Check 6

Solve the literal equation $\frac{4}{x} + \frac{2}{y} = \frac{3}{z}$ for x.

As in Chapter 1, we will rewrite our solution so that the variable we are solving for is on the left side.

EXAMPLE 7 Solve the literal equation $\frac{1}{x+y} = \frac{1}{b} + 3$ for y.

Solution

We begin by multiplying both sides of the equation by the LCD, which is $b(x+y)$.

$$\frac{1}{x+y} = \frac{1}{b} + 3$$

$$b(x+y)\frac{1}{x+y} = b(x+y)\left(\frac{1}{b} + 3\right)$$ Multiply both sides by the LCD.

$$b\cancel{(x+y)}^{1} \cdot \frac{1}{\cancel{x+y}_{1}} = \cancel{b}^{1}(x+y) \cdot \frac{1}{\cancel{b}_{1}} + b(x+y) \cdot 3$$ Distribute and divide out common factors.

$$b = x + y + 3b(x+y)$$ Multiply remaining factors.

$$b = x + y + 3bx + 3by$$ Distribute.

$$b - x - 3bx = y + 3by$$ Subtract x and $3bx$ so only the terms containing y are on the right side.

$$b - x - 3bx = y(1 + 3b)$$ Factor out the common factor y.

$$\frac{b - x - 3bx}{1 + 3b} = \frac{y\cancel{(1+3b)}^{1}}{\cancel{1+3b}_{1}}$$ Divide both sides by $1 + 3b$ to isolate y.

$$y = \frac{b - x - 3bx}{1 + 3b}$$ Rewrite with y on the left side.

Quick Check 7

Solve the literal equation $\frac{1}{x-2} + \frac{2}{y} = 5$ for x.

Building Your Study Strategy **Test Taking, 5** **Check Point Values** Not all problems on a test are assigned the same point value. Identify which problems are worth more points than others. It is important not to be in a position of having to rush through the problems that have higher point values because you didn't notice that they were worth more until you got to them at the end of the test.

EXERCISES 5.5

Vocabulary

1. A(n) __________ is an equation containing at least one rational expression.

2. To solve a rational equation, begin by multiplying both sides of the equation by the __________ of the denominators in the equation.

3. A(n) __________ of a rational equation is a solution that causes one or more of the rational expressions in the equation to be undefined.

4. A(n) __________ equation is an equation containing two or more variables.

Solve.

5. $\frac{x}{4} + \frac{7}{5} = \frac{3}{2}$

6. $\frac{x}{8} + \frac{5}{12} = \frac{17}{6}$

7. $\frac{2x}{3} + 4 = \frac{x}{8} + \frac{5}{6}$

8. $\frac{x}{4} - \frac{13}{4} = \frac{x+6}{6} - \frac{7}{2}$

9. $\frac{22}{x} - \frac{3}{4} = \frac{5}{8}$

10. $2 + \frac{27}{x} = \frac{17}{4}$

11. $\frac{9}{x} = \frac{15}{20}$

12. $5 - \frac{1}{4x} = \frac{1}{12x} - \frac{1}{3x}$

13. $x - 4 + \frac{16}{x} = 6$

14. $x - \frac{18}{x} = 3 + \frac{10}{x}$

15. $\frac{x}{9} = \frac{2}{x} - \frac{1}{3}$

16. $\frac{x}{6} + \frac{4}{x} = \frac{5}{3}$

17. $\frac{2x+1}{x-5} = \frac{x-8}{x-5}$

18. $\frac{7x+6}{x+4} = \frac{3x-10}{x+4}$

19. $\frac{5x-3}{4x-3} = \frac{13x-9}{4x-3}$

20. $\frac{x^2+8x}{x+6} = \frac{9x+42}{x+6}$

21. $3 + \frac{15x-4}{x+4} = \frac{x^2+20x}{x+4}$

22. $5 + \frac{3x-8}{x-2} = \frac{x^2+8x-54}{x-2}$

23. $\frac{3x-1}{x-7} - 4 = \frac{2x+3}{x-7}$

24. $\frac{4x-11}{3x-4} - \frac{x+24}{3x-4} = 2$

25. $x + \frac{2x+31}{x-5} = \frac{14x-39}{x-5}$

26. $x + \frac{x-25}{x+8} = \frac{6x+15}{x+8}$

27. $\frac{4}{x+7} = \frac{5}{2x-1}$

28. $\frac{8}{x+2} = \frac{18}{2x+10}$

29. $\frac{6}{x+14} = \frac{15}{8-2x}$

30. $\frac{4}{x-3} = \frac{14}{8x+3}$

31. $\frac{x+3}{4x+9} = \frac{3}{x+5}$

32. $\frac{x-6}{x+6} = \frac{3}{x+2}$

33. $\frac{x+4}{x+8} = \frac{2}{x-1}$

34. $\frac{x+7}{x+10} = \frac{5}{x-2}$

35. $\frac{x-4}{x^2+5x-24} = \frac{6}{x^2+15x+56}$

36. $\frac{x-3}{x^2+3x-4} = \frac{7}{x^2+13x+36}$

37. $\frac{x+4}{x^2-x-2} = \frac{6}{x^2-4x-5}$

38. $\frac{x-4}{x^2-3x-54} = \frac{2}{x^2-12x+27}$

39. $\frac{x+5}{x^2-10x-11} = \frac{4}{x^2-19x+88}$

40. $\frac{x+6}{x^2+8x+15} = \frac{4}{x^2+2x-15}$

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Addison-Wesley Math Tutor Center

Student's Solutions Manual

41. $\frac{2}{x+5}+\frac{3}{x+3}=\frac{x-3}{x^2+8x+15}$

42. $\frac{5}{x}+\frac{6}{x-4}=\frac{3x+4}{x^2-4x}$

43. $\frac{3}{x-2}-\frac{7}{x-8}=\frac{x-20}{x^2-10x+16}$

44. $\frac{x}{x+2}+\frac{3}{x-4}=\frac{4x+12}{x^2-2x-8}$

45. $\frac{x+4}{x-6}+\frac{3}{x+2}=\frac{x+23}{x^2-4x-12}$

46. $\frac{x+6}{x+7}-\frac{5}{x-7}=\frac{3x-7}{x^2-49}$

47. $\frac{x-3}{x+1}-\frac{7}{x-1}=\frac{x+9}{x^2-1}$

48. $\frac{x-6}{x}+\frac{9}{x-5}=\frac{11x}{x^2-5x}$

49. $\frac{x+3}{x^2+9x+20}+\frac{1}{x^2-4x-32}=\frac{2}{x^2-3x-40}$

50. $\frac{x+5}{x^2+5x-14}-\frac{4}{x^2+10x+21}=\frac{5}{x^2+x-6}$

51. $\frac{x-2}{x^2-x-20}-\frac{7}{x^2+x-12}=\frac{2}{x^2-8x+15}$

52. $\frac{x+5}{x^2-9}+\frac{8}{x^2+5x-24}=\frac{2}{x^2+11x+24}$

53. If $x = 2$ is a solution to the equation $\frac{x+1}{x+6}+\frac{1}{x+4}=\frac{?}{x^2+10x+24}$,

a) Find the constant in the missing numerator.

b) Find the other solution to the equation.

54. If $x = 5$ is a solution to the equation $\frac{x-3}{x+4}+\frac{?}{x+1}=\frac{5x+32}{x^2+5x+4}$,

a) Find the constant in the missing numerator.

b) Find the other solution to the equation.

Solve for the specified variable.

55. $W=\frac{A}{L}$ for L

56. $h=\frac{2A}{b}$ for b

57. $y=\frac{x}{3x-2}$ for x

58. $y=\frac{5x+4}{2x}$ for x

59. $\frac{x}{3r}+\frac{y}{4r}=1$ for r

60. $\frac{1}{a}-\frac{2}{b}=3$ for b

61. $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ for c

62. $1+\frac{a}{b}=\frac{c}{d}$ for b

63. $x=\frac{a+b}{a-b}$ for b

64. $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$ for R

65. $n=\frac{n_1+n_2}{n_1\cdot n_2}$ for n_1

66. $p=\frac{x_1+x_2}{n_1+n_2}$ for n_1

Writing in Mathematics

Answer in complete sentences.

67. Explain how to determine whether a solution is an extraneous solution.

68. We use the LCD when we add two rational expressions, as well as when we solve a rational equation. Explain how the LCD is used differently for these two types of problems.

69. ***Solutions Manual**** Write a solutions manual page for the following problem:

Solve $\frac{x-6}{x^2-9}+\frac{5}{x^2+4x-21}=\frac{2}{x^2+10x+21}$.

70. ***Newsletter**** Write a newsletter that explains how to solve rational equations.

***See Appendix B for details and sample answers.**

5.6 Applications of Rational Equations

Objectives

1. **Solve applied problems involving the reciprocal of a number.**
2. **Solve applied work–rate problems.**
3. **Solve applied uniform motion problems.**
4. **Solve variation problems.**

In this section, we will look at applied problems requiring the use of rational equations to solve them. We begin with problems involving reciprocals.

Solving Applied Problems Involving the Reciprocal of a Number

Objective 1 **Solve applied problems involving the reciprocal of a number.**

Example 1 The sum of the reciprocal of a number and $\frac{1}{4}$ is $\frac{1}{3}$. Find the number.

Solution

There is only one unknown in this problem, and we will let x represent it.

Unknown
Number: x

The reciprocal of this number can be written as $\frac{1}{x}$. We are told that the sum of this reciprocal and $\frac{1}{4}$ is $\frac{1}{3}$, which leads to the equation $\frac{1}{x} + \frac{1}{4} = \frac{1}{3}$.

$$\frac{1}{x} + \frac{1}{4} = \frac{1}{3} \qquad \text{The LCD is } 12x.$$

$$12x \cdot \left(\frac{1}{x} + \frac{1}{4}\right) = 12x \cdot \frac{1}{3} \qquad \text{Multiply both sides by the LCD.}$$

$$\overset{1}{12\cancel{x}} \cdot \frac{1}{\underset{1}{\cancel{x}}} + \overset{3}{\cancel{12}}x \cdot \frac{1}{\underset{1}{\cancel{4}}} = \overset{4}{\cancel{12}}x \cdot \frac{1}{\underset{1}{\cancel{3}}} \qquad \text{Distribute and divide out common factors.}$$

$$12 + 3x = 4x \qquad \text{Multiply remaining factors. The resulting equation is linear.}$$

$$12 = x \qquad \text{Subtract } 3x.$$

The unknown number is 12. The reader should verify that $\frac{1}{12} + \frac{1}{4} = \frac{1}{3}$.

Quick Check 1
The sum of the reciprocal of a number and $\frac{2}{5}$ is $\frac{1}{2}$. Find the number.

Example 2 One positive number is five larger than another positive number. If the reciprocal of the smaller number is added to eight times the reciprocal of the larger number, the sum is equal to 1. Find the two numbers.

Solution

In this problem there are two unknown numbers. If we let x represent the smaller number, then we can write the larger number as $x + 5$.

Unknowns
Smaller Number: x
Larger Number: $x + 5$

Since the unknown numbers are both positive, we must omit any solutions for which either the smaller number or larger number are not positive.

The reciprocal of the smaller number is $\frac{1}{x}$ and eight times the reciprocal of the larger number is $8 \cdot \frac{1}{x+5}$ or $\frac{8}{x+5}$. This leads to the equation $\frac{1}{x} + \frac{8}{x+5} = 1$.

$\frac{1}{x} + \frac{8}{x+5} = 1$	The LCD is $x(x+5)$.
$x(x+5)\left(\frac{1}{x} + \frac{8}{x+5}\right) = x(x+5) \cdot 1$	Multiply both sides by the LCD.
$\overset{1}{\cancel{x}}(x+5) \cdot \frac{1}{\underset{1}{\cancel{x}}} + x\overset{1}{\cancel{(x+5)}} \cdot \frac{8}{\underset{1}{\cancel{x+5}}} = x(x+5) \cdot 1$	Distribute and divide out common factors.
$x + 5 + 8x = x(x+5)$	Multiply remaining factors.
$x + 5 + 8x = x^2 + 5x$	Distribute. The resulting equation is quadratic.
$9x + 5 = x^2 + 5x$	Combine like terms.
$0 = x^2 - 4x - 5$	Collect all terms on the right side of the equation.
$0 = (x+1)(x-5)$	Factor.
$x = -1$ or $x = 5$	Set each factor equal to 0 and solve.

Since we were told that the numbers must be positive, we can omit the solution $x = -1$. We now return to the table of unknowns to find the two numbers.

Smaller Number: $x = 5$
Larger Number: $x + 5 = 5 + 5 = 10$

The two numbers are 5 and 10. The reader should verify that $\frac{1}{5} + 8 \cdot \frac{1}{10} = 1$.

***Quick Check* 2**

One positive number is 10 less than another positive number. If seven times the reciprocal of the smaller number is added to six times the reciprocal of the larger number, the sum is equal to 1. Find the two numbers.

Work–Rate Problems

Objective 2 Solve applied work–rate problems. Now we turn our attention to problems known as **work–rate problems**. These problems usually involve two or more people or objects working together to perform a job, such as Kathy and Leah painting a room together or two copy machines processing an exam. In general, the equation we will be solving corresponds to the following:

Portion of the job completed by person #1 + Portion of the job completed by person #2 = 1 (Completed job)

To determine the portion of the job completed by each person, we must know the **rate** at which the person works. If it takes Kathy 5 hours to paint a room, how much of the room could she paint in one hour? She could paint $\frac{1}{5}$ of the room in 1 hour, and this is her working rate. In general, the work rate for a person is equal to the reciprocal of the time it takes for that person to complete the whole job. If we multiply the work rate for a person by the amount of time that person has been working, then this tells us the portion of the job that the person has completed.

EXAMPLE 3 Working alone, Kathy can paint a room in 5 hours. Leah can paint the same room in only 3 hours. How long would it take the two of them to paint the room if they work together?

Solution

A good approach to any work–rate problem is to start with the following table and fill in the information.

Person	Time to complete the job alone	Work rate	Time Working	Portion of the Job Completed
Person #1				
Person #2				

- Since we know it would take Kathy 5 hours to paint the room, her work rate is $\frac{1}{5}$ room per hour. Similarly, Leah's work rate is $\frac{1}{3}$ room per hour.
- The unknown in this problem is the amount of time they will be working together, which we will represent by the variable t.
- Finally, to determine the portion of the job completed by each person, we multiply the person's work rate by the time they have been working.

Person	Time to complete the job alone	Work rate	Time Working	Portion of the Job Completed
Kathy	5 hours	$\frac{1}{5}$	t	$\frac{t}{5}$
Leah	3 hours	$\frac{1}{3}$	t	$\frac{t}{3}$

To find the equation we need to solve, we add the portion of the room painted by Kathy to the portion of the room painted by Leah and set this sum equal to 1. The equation for this problem is $\frac{t}{5} + \frac{t}{3} = 1$.

***Quick Check* 3**

Marco's new printer, working alone, can print a complete set of brochures in 15 minutes. His old printer can print the set of brochures in 40 minutes. How long would it take the two printers to print the set of brochures if they work together?

$$\frac{t}{5} + \frac{t}{3} = 1 \qquad \text{The LCD is 15.}$$

$$15\left(\frac{t}{5} + \frac{t}{3}\right) = 15 \cdot 1 \qquad \text{Multiply both sides by the LCD.}$$

$$\overset{3}{\cancel{15}} \cdot \frac{t}{\underset{1}{\cancel{5}}} + \overset{5}{\cancel{15}} \cdot \frac{t}{\underset{1}{\cancel{3}}} = 15 \cdot 1 \qquad \text{Distribute and divide out common factors.}$$

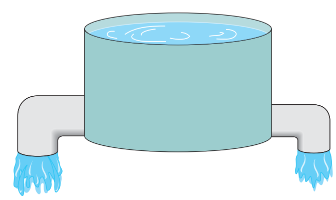
$$3t + 5t = 15 \qquad \text{Multiply remaining factors.}$$

$$8t = 15 \qquad \text{Combine like terms.}$$

$$t = \frac{15}{8} \quad \text{or} \quad 1\frac{7}{8} \qquad \text{Divide both sides by 8.}$$

It would take them $1\frac{7}{8}$ hours to paint the room if they worked together. To convert the answer to hours and minutes we multiply $\frac{7}{8}$ of an hour by 60 minutes per hour, which is $52\frac{1}{2}$ minutes. It would take them approximately one hour and $52\frac{1}{2}$ minutes. The check of this solution is left to the reader.

EXAMPLE 4 Two drainpipes, working together, can drain a pool in 3 hours. Working alone, it would take the smaller pipe 8 hours longer than it would take the larger pipe to drain the pool. How long would it take the smaller pipe alone to drain the pool?



Solution

In this example we know the amount of time it would take the pipes to drain the pool if they were working together, but we do not know how long it would take each pipe working alone. If we let t represent the time, in hours, that it takes the larger pipe to drain the pool, then the smaller pipe takes $t + 8$ hours to drain the pool. We will begin with a table.

	Time to Complete the Job Alone	Work Rate	Time Working	Portion of the Job Completed
Smaller pipe	$t + 8$ hours	$\frac{1}{t+8}$	3 hours	$\frac{3}{t+8}$
Larger pipe	t hours	$\frac{1}{t}$	3 hours	$\frac{3}{t}$

When we add the portion of the pool drained by the smaller pipe in 3 hours to the portion of the pool drained by the larger pipe, the sum will equal 1. The equation is $\frac{3}{t+8} + \frac{3}{t} = 1$.

$$\frac{3}{t+8}+\frac{3}{t}=1$$ The LCD is $t(t+8)$.

$$t(t+8)\cdot\left(\frac{3}{t+8}+\frac{3}{t}\right)=t(t+8)\cdot 1$$ Multiply both sides by the LCD.

$$t\cancel{(t+8)}^{1}\cdot\frac{3}{\cancel{t+8}_{1}}+\cancel{t}^{1}(t+8)\cdot\frac{3}{\cancel{t}_{1}}=t(t+8)\cdot 1$$ Distribute and divide out common factors.

$$3t+3(t+8)=t(t+8)$$ Multiply remaining factors.

$$3t+3t+24=t^2+8t$$ Multiply. The resulting equation is quadratic.

$$6t+24=t^2+8t$$ Combine like terms.

$$0=t^2+2t-24$$ Collect all terms on the right side of the equation.

$$0=(t+6)(t-4)$$ Factor.

$$t=-6 \quad\text{or}\quad t=4$$ Set each factor equal to 0 and solve.

Since the time spent by each pipe must be positive, we may immediately omit the solution $t=-6$. The reader may check the solution $t=4$ by verifying that $\frac{3}{4+8}+\frac{3}{4}=1$. We now use $t=4$ to find the amount of time it would take the smaller pipe to drain the pool. Since $t+8$ represents the amount of time it would take for the smaller pipe to drain the pool, it would take the smaller pipe $4+8$ or 12 hours to drain the pool.

Quick Check 4
Two drainpipes, working together, can drain a tank in 4 hours. Working alone, it would take the smaller pipe 6 hours longer than it would take the larger pipe to drain the tank. How long would it take the smaller pipe alone to drain the tank?

Uniform Motion Problems

Objective 3 **Solve applied uniform motion problems.** We now turn our attention to uniform motion problems. Recall that if an object is moving at a constant rate of speed for a certain amount of time, then the distance traveled by the object is equal to the product of its rate of speed and the length of time it traveled. The formula we used earlier in the text is rate · time = distance or $r\cdot t=d$. If we solve this formula for the time traveled, we have

$$\text{time}=\frac{\text{distance}}{\text{rate}}\quad\text{or}\quad t=\frac{d}{r}$$

EXAMPLE 5 Drake drove 30 miles, one way, to make a sales call. His second sales call was 35 miles away from the first sales call, and he drove 10 miles per hour faster than he did on his way to his first sales call. If the driving time was 1 hour for the entire trip, find Drake's driving speed on the way to his second sales call.

Solution

We know that the distance traveled to the first sales call is 30 miles, and the distance traveled to the second sales call is 35 miles. Drake's speed on the way to the second sales call was 10 miles per hour faster than it was on the way to the first sales call. We will let r represent his rate of speed on the way to the first sales call, and we can represent his rate of speed on the way to the second sales call as $r+10$. To find an expression for the time spent on each part of the trip, we divide the distance by the rate.

We can summarize this information in a table:

	Distance (d)	Rate (r)	Time (t)
To First Sales Call	30 miles	r	$\frac{30}{r}$
To Second Sales Call	35 miles	$r+10$	$\frac{35}{r+10}$

The equation we need to solve comes from the fact that the driving time on the way to the first sales call plus the driving time on the way to the second sales call is equal to 1 hour. If we add the time spent on the way to the first sales call $\left(\frac{30}{r}\right)$ to the time spent on the way the second sales call $\left(\frac{35}{r+10}\right)$, this will be equal to 1. The equation we need to solve is $\frac{30}{r}+\frac{35}{r+10}=1$.

$$\frac{30}{r}+\frac{35}{r+10}=1 \qquad \text{The LCD is } r(r+10).$$

$$r(r+10)\cdot\left(\frac{30}{r}+\frac{35}{r+10}\right)=r(r+10)\cdot 1 \qquad \text{Multiply both sides by the LCD.}$$

$$\overset{1}{\cancel{r}}(r+10)\cdot\frac{30}{\underset{1}{\cancel{r}}}+r\overset{1}{\cancel{(r+10)}}\cdot\frac{35}{\underset{1}{\cancel{r+10}}}=r(r+10)\cdot 1 \qquad \text{Distribute and divide out common factors.}$$

$$30(r+10)+35r=r(r+10) \qquad \text{Multiply remaining factors.}$$

$$30r+300+35r=r^2+10r \qquad \text{Multiply.}$$

$$65r+300=r^2+10r \qquad \text{Combine like terms.}$$

$$0=r^2-55r-300 \qquad \text{Collect all terms on the right side of the equation by subtracting } 65r \text{ and } 300.$$

$$0=(r-60)(r+5) \qquad \text{Factor.}$$

$$r=60 \quad \text{or} \quad r=-5 \qquad \text{Set each factor equal to 0 and solve.}$$

We omit the solution $r=-5$, as Drake's speed cannot be negative. The expression for Drake's driving speed on the way to the second sales call is $r+10$, so his speed on the way to the second sales call was $60+10$ or 70 miles per hour. The reader can check the solution by verifying that $\frac{30}{60}+\frac{35}{70}=1$.

***Quick Check* 5**

Liana drove 240 miles to pick up a friend and then returned home. On the way home, she drove 20 miles per hour faster than she did on her way to pick up her friend. If the total driving time for the trip was 7 hours for the entire trip, find Liana's driving speed on the way home.

EXAMPLE 6 Stephanie took her kayak to the Kaweah River, which flows downstream at a rate of 2 kilometers per hour. She paddled 15 km upstream, and then paddled downstream to her starting point. If this round-trip took a total of 4 hours, find the speed that Stephanie can paddle in still water.

Solution

We will let r represent the speed that Stephanie can paddle in still water. Since the current of the river is 2 kilometers per hour, Stephanie's kayak travels at a speed of $r-2$ kilometers per hour when she is paddling upstream. This is because the current is pushing against the kayak. Stephanie travels at a speed of $r+2$ kilometers per hour when she is paddling downstream, as the current is flowing in the same direction as the kayak. The equation we will solve includes the time spent paddling upstream and downstream. To find expressions in terms of r for the time spent in each direction, we divide the distance (15 km) by the rate of speed. Here is a table containing the relevant information.

	Distance	Rate	Time
Upstream	15 km	$r - 2$	$\frac{15}{r-2}$
Downstream	15 km	$r + 2$	$\frac{15}{r+2}$

We are told that the time needed to make the round-trip is 4 hours. In other words, the time spent paddling upstream plus the time spent paddling downstream is equal to 4 hours, or $\frac{15}{r-2} + \frac{15}{r+2} = 4$.

$$\frac{15}{r-2} + \frac{15}{r+2} = 4$$

The LCD is $(r-2)(r+2)$.

$$(r-2)(r+2)\cdot\left(\frac{15}{r-2} + \frac{15}{r+2}\right) = (r-2)(r+2)\cdot 4$$

Multiply both sides by the LCD.

$$\overset{1}{\cancel{(r-2)}}(r+2)\cdot\frac{15}{\underset{1}{\cancel{r-2}}} + (r-2)\overset{1}{\cancel{(r+2)}}\frac{15}{\underset{1}{\cancel{r+2}}} = (r-2)(r+2)\cdot 4$$

Distribute and divide out common factors.

$$15(r+2) + 15(r-2) = 4(r-2)(r+2)$$

Multiply remaining factors.

$$15r + 30 + 15r - 30 = 4r^2 - 16$$

Multiply.

$$30r = 4r^2 - 16$$

Combine like terms.

$$0 = 4r^2 - 30r - 16$$

Collect all terms on the right side of the equation by subtracting $30r$

$$0 = 2(2r^2 - 15r - 8)$$

Factor out the common factor 2.

$$0 = 2(2r+1)(r-8)$$

Factor.

$$r = -\frac{1}{2} \quad \text{or} \quad r = 8$$

Set each variable factor equal to 0 and solve.

We omit the negative solution, as the speed of the kayak in still water must be positive. Stephanie's kayak travels at a speed of 8 kilometers per hour in still water. The reader can check this solution by verifying that $\frac{15}{8-2} + \frac{15}{8+2} = 4$.

***Quick Check* 6**

Jacob took his canoe to a river that flows downstream at a rate of 2 miles per hour. He paddled 24 miles downstream, and then returned back to the camp he started from. If the round-trip took him 7 hours, find the speed that Jacob can paddle in still water.

Variation

Objective 4 **Solve applied variation problems.** In the remaining examples, we will investigate the concept of **variation** between two or more quantities. Two quantities are said to **vary directly** if an increase in one quantity produces a proportional increase in the other quantity, and a decrease in one quantity produces a proportional decrease in the other quantity. For example, suppose that you have a part-time job that pays by the hour. The hours you work in a week and the amount of money you earn (before taxes) vary directly. As the hours you work increase, the amount of money you earn increases by the same factor. If there is a decrease in the number of hours you work, the amount of money that you earn decreases by the same factor.

Direct Variation

If a quantity y varies directly as a quantity x, then the two quantities are related by the equation

$$y = kx$$

where k is called the **constant of variation**.

For example, if you are paid \$12 per hour at your part-time job, then the amount of money you earn (y) and the number of hours you work (x) are related by the equation $y = 12x$. The value of k in this situation is 12, and it tells us that each time x increases by 1 hour, y increases by \$12.

Example 7 y varies directly as x. If $y = 54$ when $x = 9$, find y when $x = 17$.

Solution

We begin by finding k. Using $y = 54$ when $x = 9$, we can use the equation $54 = k \cdot 9$ to find k.

$54 = k \cdot 9$ Substitute 54 for y and 9 for x into $y = kx$.
$6 = k$ Divide both sides by 9.

Now we use this value of k to find y when $x = 17$.

$y = 6 \cdot 17$ Substitute 6 for k and 17 for x into $y = kx$.
$y = 102$ Multiply.

Example 8 The distance a train travels varies directly as the time it is traveling. If a train can travel 399 miles in 7 hours, how far can it travel in 12 hours?

Solution

The first step in a variation problem is to find k. We will let y represent the distance traveled and x represent the time. To find the variation constant, we will substitute the related information (399 miles in 7 hours) into the equation for direct variation.

$399 = k \cdot 7$ Substitute 399 for y and 7 for x into $y = kx$.
$57 = k$ Divide both sides by 7.

Now we will use this variation constant to find the distance traveled in 12 hours.

$y = 57 \cdot 12$ Substitute 57 for k and 12 for x into $y = kx$.
$y = 684$ Multiply.

The train travels 684 miles in 12 hours.

***Quick Check* 7** **The tuition a college student pays varies directly as the number of units the student is taking. If a student pays \$720 to take 15 units in a semester, how much would a student pay to take 12 units?**

In some cases, an increase in one quantity produces a *decrease* in another quantity. In this case we say that the quantities **vary inversely**. For example, suppose that you and some of your friends are going to buy a birthday gift for someone, splitting the cost equally. As the number of people who are contributing increases, the cost for each person decreases. The cost for each person varies inversely as the number of people contributing.

Inverse Variation

If a quantity y varies inversely as a quantity x, then the two quantities are related by the equation

$$y = \frac{k}{x}$$

where k is the constant of variation.

EXAMPLE 9 y varies inversely as x. If $y = 24$ when $x = 5$, find y when $x = 8$.

Solution

We begin by finding k.

$24 = \frac{k}{5}$ Substitute 24 for y and 5 for x into $y = \frac{k}{x}$.

$120 = k$ Multiply both sides by 5.

Now we use this value of k to find y when $x = 8$.

$y = \frac{120}{8}$ Substitute 120 for k and 8 for x into $y = \frac{k}{x}$.

$y = 15$ Divide.

EXAMPLE 10 The time required to drive from Cincinnati to Cleveland varies inversely as the average speed of the car. If it takes 4 hours to make the drive at 60 miles per hour, how long would it take to make the drive at 80 miles per hour?

Solution

Again, we begin by finding k. We will let y represent the time required to drive from Cincinnati to Cleveland and x represent the average speed of the car. To find k, we will substitute the related information (4 hours to make the trip at 60 miles per hour) into the equation for inverse variation.

$4 = \frac{k}{60}$ Substitute 4 for y and 60 for x into $y = \frac{k}{x}$.

$240 = k$ Multiply both sides by 60.

Now we will use this variation constant to find the time required to drive from Cincinnati to Cleveland at 80 miles per hour.

$$y = \frac{240}{80} \quad \text{Substitute 240 for } k \text{ and 80 for } x \text{ into } y = \frac{k}{x}.$$

$$y = 3 \quad \text{Divide.}$$

It would take 3 hours to drive from Cincinnati to Cleveland at 80 miles per hour.

Quick Check 8 **The time required to complete a cycling race varies inversely as the average speed of the cyclist. If it takes 15 minutes to complete the race at an average speed of 40 kilometers per hour, how long would it take to complete the race at an average speed of 30 kilometers per hour?**

Often a quantity varies depending on two or more variables. A quantity y is said to **vary jointly** as two quantities x and z if it varies directly as the product of these two quantities. The equation in such a case is $y = kxz$.

EXAMPLE 11 y varies jointly as x and the square of z. If $y = 1800$ when $x = 9$ and $z = 5$, find y when $x = 15$ and $z = 10$.

Solution

We begin by finding k, using the equation $y = kxz^2$.

$$1800 = k \cdot 9 \cdot 5^2 \quad \text{Substitute 1800 for } y\text{, 9 for } x\text{, and 5 for } z \text{ into } y = kxz^2.$$

$$1800 = k \cdot 225 \quad \text{Simplify.}$$

$$8 = k \quad \text{Divide both sides by 225.}$$

Now we use this value of k to find y when $x = 15$ and $z = 10$.

$$y = 8 \cdot 15 \cdot 10^2 \quad \text{Substitute 8 for } k\text{, 15 for } x\text{, and 10 for } z \text{ into } y = kxz^2.$$

$$y = 12{,}000 \quad \text{Simplify.}$$

Quick Check 9 **y varies jointly as x and the square root of z. If $y = 600$ when $x = 40$ and $z = 36$, find y when $x = 200$ and $z = 25$.**

Building Your Study Strategy **Test Taking, 6** **Easy Problems First?** When taking a test, work on the easiest problems first, saving the more difficult problems for later. One benefit to this approach is that you will gain confidence as you progress through the test, so you are confident when you attempt to solve a difficult problem. Students who struggle on a problem on the test will lose confidence, and this may cause them to miss later problems that they know how to do. Also, if you complete the easier problems quickly first, you will save time to work on the few difficult problems. One exception to this practice is to begin the test by first attempting the most difficult type of problem for you. For example, if a particular type of word problem has been difficult for you to solve throughout your preparation for the exam, you may want to focus on how to solve that type of problem just before arriving for the test. With the solution of this type of problem committed to your short-term memory, your best opportunity for success comes at the beginning of the test prior to solving any other problems.

Exercises 5.6

Vocabulary

1. The __________ of a number x is $\frac{1}{x}$.

2. The __________ of a person is the amount of work he or she does in one unit of time.

3. If an object is moving at a constant rate of speed for a certain amount of time, then the length of time it traveled is equal to the distance traveled by the object __________ by its rate of speed.

4. If a canoe can travel at a rate of r miles per hour in still water and a river's current is c miles per hour, then the rate of speed that the canoe travels at while moving upstream is given by the expression __________.

5. Two quantities are said to __________ if an increase in one quantity produces a proportional increase in the other quantity.

6. Two quantities are said to __________ if an increase in one quantity produces a proportional decrease in the other quantity.

7. The sum of the reciprocal of a number and $\frac{2}{3}$ is $\frac{11}{12}$. Find the number.

8. The sum of the reciprocal of a number and $\frac{5}{6}$ is $\frac{23}{24}$. Find the number.

9. The sum of 5 times the reciprocal of a number and $\frac{7}{2}$ is $\frac{31}{6}$. Find the number.

10. The sum of 7 times the reciprocal of a number and $\frac{3}{4}$ is $\frac{4}{3}$. Find the number.

11. The difference of the reciprocal of a number and $\frac{3}{8}$ is $-\frac{7}{40}$. Find the number.

12. The difference of the reciprocal of a number and $\frac{3}{14}$ is $\frac{2}{7}$. Find the number.

13. One positive number is four larger than another positive number. If three times the reciprocal of the smaller number is added to two times the reciprocal of the larger number, the sum is equal to 1. Find the two numbers.

14. One positive number is six less than another positive number. If eight times the reciprocal of the smaller number is added to six times the reciprocal of the larger number, the sum is equal to 1. Find the two numbers.

15. One positive number is four less than another positive number. If the reciprocal of the smaller number is added to five times the reciprocal of the larger number, the sum is equal to $\frac{13}{24}$. Find the two numbers.

16. One positive number is four larger than another positive number. If five times the reciprocal of the smaller number is added to three times the reciprocal of the larger number, the sum is equal to $\frac{17}{15}$. Find the two numbers.

17. One copy machine can run off a set of copies in 30 minutes. A newer machine can do the same job in 20 minutes. How long would it take the two machines, working together, to make all of the necessary copies?

18. Dylan can mow a lawn in 30 minutes, while Alycia takes 45 minutes to mow the same lawn. If Dylan and Alycia work together, using two lawn mowers, how long would it take them to mow the lawn?

19. Dianne can completely weed her vegetable garden in 75 minutes. Her friend Gabby can do the same task in 90 minutes. If they worked together, how long would it take Dianne and Gabby to weed the vegetable garden?

20. One hose can fill a 40,000-gallon swimming pool in 16 hours. A hose from the neighbor's house can fill a swimming pool of that size in 20 hours. If the two hoses run at the same time, how long would they take to fill the swimming pool?

21. Anita can paint a room in 2 hours. Barbara can do the same job in 3 hours, while it would take Carol 4 hours to paint the entire room. If the three friends work together, how long would it take them to paint an entire room?

22. A company has printed out 1000 surveys to put in preaddressed envelopes and mail out. Bill can fill all of the envelopes in 8 hours, Terrell can do the job in 10 hours, and Jerry would take 15 hours to do the job. If all three work together, how long will it take them to stuff the 1000 envelopes?

23. Two drainpipes, working together, can drain a pool in 10 hours. Working alone, it would take the smaller pipe 15 hours longer than it would take the larger pipe to drain the pool. How long would it take the smaller pipe alone to drain the pool?

24. Jacqui and Jill, working together, can paint the interior of a house in 6 hours. If Jill were working alone, it would take her 5 hours longer than it would take Jacqui to paint the interior of a house. How long would it take Jacqui to paint the interior of a house by herself?

25. An electrician and his assistant can wire a house in 4 hours. If they worked alone, it would take the assistant 6 hours longer than the electrician to wire the house. How long would it take the assistant to wire the house?

26. Wayne and Ed can clean an entire building in 2 hours. Wayne can clean the entire building by himself in 3 hours less time than Ed can. How long would it take Wayne to clean the building by himself?

27. Two pipes, working together, can drain a tank in 6 minutes. The smaller pipe, working alone, would take 3 times as long as the larger pipe to drain the tank. How long would it take the smaller pipe, working alone, to drain the tank?

28. Carolyn and Laura can muck out their stable in 9 minutes. Working alone, it would take Laura three times as long as it would take Carolyn to muck out the stable. How long would it take Laura, working alone, to muck out the stable?

29. A college has two printers available to print out class rosters for the first day of classes. The older printer takes 10 hours longer than the newer printer to print out a complete set of class rosters, so the dean of registration decides to use the newer printer. After 3 hours the newer printer breaks down, so the dean must switch to the older printer. It takes the older printer an additional 8 hours to complete the job. How long would it have taken the older printer to print out a complete set of class rosters?

30. A bricklayer's apprentice takes 10 hours longer than the bricklayer to make a fireplace. The apprentice worked alone on a fireplace for 5 hours, after which the bricklayer began to help. It took 2 more hours for the pair to finish the fireplace. How long would it take the apprentice to make a fireplace on his own?

31. Ross is training for a triathlon. He rides his bicycle at a speed that is 20 miles per hour faster than his running speed. If Ross can cycle 70 miles in the same amount of time that it takes him to run 12 miles, what is Ross's running speed?

32. Ariel rides her bicycle at a speed that is 10 miles per hour faster than the speed that Sharon rides her bicycle. Ariel can ride 75 miles in the same amount of time that it takes for Sharon to ride 50 miles. How fast does Ariel ride her bicycle?

33. Jared is training to run a marathon. Today he ran 14 miles in 2 hours. After running the first 8 miles at a certain speed, he increased his speed by 3 miles per hour for the remaining 6 miles. Find the speed at which Jared ran the last 6 miles.

34. Cecilia drove 195 miles home from college to spend the weekend with her family. After the first 105 miles, she increased her driving speed by 12 miles per hour. If it took Cecilia 3 hours to get home, find her speed for the first 105 miles of the trip.

35. A salmon is swimming in a river that is flowing downstream at a speed of 8 kilometers per hour. The salmon can swim 8 kilometers upstream in the same time that it would take to swim 12 kilometers downstream. What is the speed of the salmon in still water?

36. Janet is swimming in a river that flows downstream at a speed of 0.5 meters per second. It takes her the same amount of time to swim 300 meters upstream as it does to swim 500 meters downstream. Find Janet's swimming speed (in meters per second) in still water.

37. An airplane flies 300 miles with a 40 mile per hour tailwind, and then flies 300 miles back into the 40 mile per hour wind. If the time for the round-trip was 4 hours, find the speed of the airplane in calm air.

38. Selma is kayaking in a river that flows downstream at a rate of 1 mile per hour. Selma paddles 9 miles downstream and then turns around and paddles 10 miles upstream, and the trip takes 4 hours.

a) How fast can Selma paddle in still water?

b) Selma is now 1 mile upstream of her starting point. How many minutes will it take her to paddle back to her starting point?

39. y varies directly as x. If $y = 54$ when $x = 9$, find y when $x = 21$.

40. y varies directly as x. If $y = 91$ when $x = 7$, find y when $x = 18$.

41. y varies directly as x. If $y = 40$ when $x = 15$, find y when $x = 27$.

42. y varies directly as x. If $y = 48$ when $x = 84$, find y when $x = 35$.

43. y varies inversely as x. If $y = 12$ when $x = 14$, find y when $x = 8$.

44. y varies inversely as x. If $y = 20$ when $x = 24$, find y when $x = 30$.

45. y varies inversely as x. If $y = 10$ when $x = 9$, find y when $x = 20$.

46. y varies inversely as x. If $y = 50$ when $x = 24$, find y when $x = 250$.

47. y varies directly as the square of x. If $y = 72$ when $x = 3$, find y when $x = 6$.

48. y varies inversely as the square of x. If $y = 10$ when $x = 8$, find y when $x = 2$.

49. y varies jointly as x and z. If $y = 630$ when $x = 6$ and $z = 7$, find y when $x = 5$ and $z = 12$.

50. y varies directly as x and inversely as z. If $y = 18$ when $x = 12$ and $z = 8$, find y when $x = 55$ and $z = 15$.

51. Stuart's gross pay varies directly as the number of hours he works. In a week that he worked 24 hours, his gross pay was $258. What would Stuart's gross pay be if he worked 36 hours?

52. The height that a ball bounces varies directly as the height it is dropped from. If a ball dropped from a height of 48 inches bounces 20 inches, how high would the ball bounce if it were dropped from a height of 72 inches?

53. **Ohm's law.** In a circuit the electric current (in amperes) varies directly as the voltage. If the current is 8 amperes when the voltage is 24 volts, find the current when the voltage is 9 volts.

54. **Hooke's law.** The distance that a hanging object stretches a spring varies directly as the mass of the object. If a 5-kilogram weight stretches a spring by 32 centimeters, how far would a 2-kilogram weight stretch the spring?

55. The amount of money that each person must contribute to buy a retirement gift for a coworker varies inversely as the number of people contributing. If 16 people would each have to contribute $15, how much would each person have to contribute if there were 24 people?

56. The maximum load that a wooden beam can support varies inversely as its length. If a beam that is 6 feet long can support 900 pounds, what is the maximum load that can be supported by a beam that is 8 feet long?

57. In a circuit the electric current (in amperes) varies inversely as the resistance (in ohms). If the current is 30 amperes when the resistance is 3 ohms, find the current when the resistance is 5 ohms.

58. **Boyle's law.** The volume of a gas varies inversely as the pressure upon it. The volume of a gas is 128 cubic centimeters when it is under a pressure of 50 kilograms per square centimeter. Find the volume when the pressure is reduced to 40 kilograms per square centimeter.

59. The illumination of an object varies inversely as the square of its distance from the source of light. If a light source provides an illumination of 36 foot-candles at a distance of 8 feet, find the illumination at a distance of 24 feet. (One foot-candle is the amount of illumination produced by a standard candle at a distance of one foot.)

60. The illumination of an object varies inversely as the square of its distance from the source of light. If a light source provides an illumination of 75 foot-candles at a distance of 6 feet, find the illumination at a distance of 30 feet.

Writing in Mathematics

Answer in complete sentences.

61. Write a work–rate word problem that can be solved by the equation $\frac{3}{t+6} + \frac{10}{t} = 1$. Explain how you created your problem.

62. Write a uniform motion word problem that can be solved by the equation $\frac{3}{r-4} + \frac{3}{r+4} = 1$. Explain how you created your problem.

63. ***Solutions Manual**** Write a solutions manual page for the following problem:

James drives 150 miles to his job each day. Due to a rainstorm on his trip home, he had to drive 25 miles per hour slower than he drove on his way to work, and it took him 1 hour longer to get home than it did to get to work. How fast was James driving on his way home?

64. ***Newsletter**** Write a newsletter that explains how to solve work–rate problems.

***See Appendix B for details and sample answers.**

5.7 Division of Polynomials

Objectives

1. **Divide a monomial by a monomial.**
2. **Divide a polynomial by a monomial.**
3. **Divide a polynomial by a polynomial using long division.**
4. **Use placeholders when dividing a polynomial by a polynomial.**

A rational expression is a quotient of two polynomials. In this section, we will explore another method for simplifying a quotient of two polynomials by using division. This technique is particularly useful when the numerator and denominator do not contain common factors.

Dividing Monomials by Monomials

Objective 1 Divide a monomial by a monomial. We will begin by reviewing how to divide a monomial by another monomial, such as $\frac{4x^5}{2x^2}$ or $\frac{30a^5b^4}{6ab^2}$.

Dividing a Monomial by a Monomial

To divide a monomial by another monomial, we divide the coefficients first. Then we divide the variables using the quotient rule $\frac{x^m}{x^n} = x^{m-n}$, assuming that no variable in the denominator is equal to 0.

EXAMPLE 1 Divide $\frac{20x^8}{5x^3}$. (Assume $x \neq 0$.)

Solution

$$\frac{20x^8}{5x^3} = 4x^{8-3}$$ Divide coefficients. Subtract exponents for *x*.

$$= 4x^5$$ Simplify the exponent.

We can check our quotients by using multiplication. If $\frac{20x^8}{5x^3} = 4x^5$, then we know that $5x^3 \cdot 4x^5$ should be equal to $20x^8$.

Check:

$$5x^3 \cdot 4x^5 = 20x^{3+5}$$ Multiply coefficients. Keep the base *x*, add the exponents.

$$= 20x^8$$ Simplify the exponent.

Our quotient of $4x^5$ checks.

Quick Check **1** Divide $\frac{42x^{10}}{7x^2}$. (Assume $x \neq 0$.)

EXAMPLE 2 Divide $\frac{32x^7y^6}{4xy^6}$. (Assume $x, y \neq 0$.)

Solution

When there is more than one variable, as in this example, we divide the coefficients and then divide the variables one at a time.

$$\frac{32x^7y^6}{4xy^6} = 8x^6y^0 \quad \text{Divide coefficients, subtract exponents.}$$
$$= 8x^6 \quad \text{Rewrite without } y \text{ as a factor. } (y^0 = 1)$$

Quick Check 2

Divide $\frac{56x^8y^5}{4x^3y^2}$. (Assume $x, y \neq 0$.)

Dividing Polynomials by Monomials

Objective 2 **Divide a polynomial by a monomial.** Now we move on to dividing a polynomial by a monomial, such as $\frac{20x^5 - 30x^3 - 35x^2}{5x^2}$.

Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 3 Divide $\frac{32x^2 + 40x - 4}{4}$.

Solution

We will divide each term in the numerator by 4.

$$\frac{32x^2 + 40x - 4}{4} = \frac{32x^2}{4} + \frac{40x}{4} - \frac{4}{4} \quad \text{Divide each term in the numerator by 4.}$$
$$= 8x^2 + 10x - 1 \quad \text{Divide.}$$

If $\frac{32x^2 + 40x - 4}{4} = 8x^2 + 10x - 1$, then $4(8x^2 + 10x - 1)$ should equal $32x^2 + 40x - 4$. We can use this to check our work.

Check:

$$4(8x^2 + 10x - 1) = 4 \cdot 8x^2 + 4 \cdot 10x - 4 \cdot 1 \quad \text{Distribute.}$$
$$= 32x^2 + 40x - 4 \quad \text{Multiply.}$$

Our quotient of $8x^2 + 10x - 1$ checks.

Quick Check 3

Divide $\frac{6x^2 - 30x - 18}{6}$.

EXAMPLE 4 Divide $\dfrac{20x^5 - 30x^3 - 35x^2}{5x^2}$. (Assume $x \neq 0$.)

Quick Check 4
Divide $\dfrac{8x^9 - 24x^7 + 120x^4}{8x^3}$.
(Assume $x \neq 0$.)

Solution

In this example we will divide each term in the numerator by $5x^2$.

$$\frac{20x^5 - 30x^3 - 35x^2}{5x^2} = \frac{20x^5}{5x^2} - \frac{30x^3}{5x^2} - \frac{35x^2}{5x^2}$$ Divide each term in the numerator by $5x^2$.

$$= 4x^3 - 6x - 7$$ Divide.

Dividing a Polynomial by a Polynomial (Long Division)

Objective 3 **Divide a polynomial by another polynomial using long division.**
To divide a polynomial by another polynomial containing at least two terms, we use a procedure similar to long division. Before outlining this procedure, let's review some of the terms associated with long division.

Quotient

$$\text{Divisor} \longrightarrow 2\overset{\displaystyle 3}{\overline{)6}}$$

Dividend

Suppose that we were asked to divide $\dfrac{x^2 - 14x + 48}{x - 8}$. The polynomial in the numerator is the dividend and the polynomial in the denominator is the divisor. We may rewrite this division as $x - 8\overline{)x^2 - 14x + 48}$. We must be sure to write both the divisor and the dividend in descending order. (The term with the highest degree goes first, then the term with the next highest degree, and so on.) We perform the division using the following steps.

Division by a Polynomial

1. Divide the term in the dividend with the highest degree by the term in the divisor with the highest degree. Add this result to the quotient.
2. Multiply the monomial obtained in step 1 by the divisor, writing the result underneath the dividend. (Align like terms vertically.)
3. Subtract the product obtained in step 2 from the dividend. (Recall that to subtract a polynomial from another polynomial, we change the signs of its terms and then combine like terms.)
4. Repeat steps 1–3 with the result of step 3 as the new dividend. Keep repeating this procedure until the degree of the new dividend is less than the degree of the divisor.

EXAMPLE 5 Divide $\frac{x^2 - 14x + 48}{x - 8}$.

Solution

We begin by writing $x - 8\overline{)x^2 - 14x + 48}$. We divide the term in the dividend with the highest degree (x^2) by the term in the divisor with the highest degree (x). Since $\frac{x^2}{x} = x$, we will write x in the quotient and multiply x by the divisor $x - 8$, writing this product underneath the dividend.

$$\begin{array}{r} x \\ x - 8\overline{)x^2 - 14x + 48} \\ x^2 - 8x \end{array}$$

Multiply $x(x - 8)$.

To subtract, we may change the signs of the second polynomial and then combine like terms.

$$\begin{array}{r} x \\ x - 8\overline{)\;x^2 - 14x + 48} \\ \underline{\overset{-}{\;}x^2 \overset{+}{\cancel{-}} 8x \downarrow} \\ -6x + 48 \end{array}$$

Change the signs and combine like terms.

We now begin the process again by dividing $-6x$ by x, which equals -6. Multiply -6 by $x - 8$ and subtract.

$$\begin{array}{r} x - 6 \\ x - 8\overline{)\;x^2 - 14x + 48} \\ \underline{\overset{-}{\;}x^2 \overset{+}{\cancel{-}} 8x \downarrow} \\ -6x + 48 \\ \underline{\overset{+}{\cancel{-}}\;6x \overset{-}{\cancel{+}} 48} \\ 0 \end{array}$$

Multiply -6 by $x - 8$. Subtract the product.

The remainder of 0 tells us that we are through, since its degree is less than the degree of the divisor $x - 8$. The expression written above the division box $(x - 6)$ is the quotient.

$$\frac{x^2 - 14x + 48}{x - 8} = x - 6.$$

We can check our work by multiplying the quotient $(x - 6)$ by the divisor $(x - 8)$, which should equal the dividend $(x^2 - 14x + 48)$.

Quick Check 5

Divide $\frac{x^2 + 4x - 21}{x + 7}$.

Check:

$$\begin{aligned} (x - 6)(x - 8) &= x^2 - 8x - 6x + 48 \\ &= x^2 - 14x + 48 \end{aligned}$$

Our division checks; the quotient is $x - 6$.

In the previous example, the remainder of 0 also tells us that $x - 8$ divides into $x^2 - 14x + 48$ evenly, so $x - 8$ is a **factor** of $x^2 - 14x + 48$. The quotient, $x - 6$, is also a factor of $x^2 - 14x + 48$.

In the next example, we will learn how to write a quotient when there is a nonzero remainder.

EXAMPLE 6 Divide $x^2 - 13x + 33$ by $x - 4$.

Solution

Since the divisor and dividend are already written in descending order, we may begin to divide.

$$\begin{array}{r l} x - 9 & \\ x - 4\overline{)\,x^2 - 13x + 33} & \\ \underline{{}^{-}x^2 \overset{+}{\cancel{-}}\ 4x \quad \downarrow} & \text{Multiply } x \text{ by } x - 4 \text{ and subtract.} \\ -\ 9x + 33 & \\ \underline{\overset{+}{\cancel{-}}\ 9x \overset{-}{\cancel{+}}\ 36} & \text{Multiply } -9 \text{ by } x - 4 \text{ and subtract.} \\ -3 & \end{array}$$

Quick Check 6
Divide $\dfrac{x^2 + 5x - 5}{x + 7}$.

The remainder is -3. After the quotient, we write a fraction with the remainder in the numerator and the divisor in the denominator. Since the remainder is negative, we subtract this fraction from the quotient. If the remainder had been positive, we would have added this fraction to the quotient.

$$\frac{x^2 - 13x + 33}{x - 4} = x - 9 - \frac{3}{x - 4}$$

EXAMPLE 7 Divide $6x^2 - 13x + 13$ by $3x + 4$.

Solution

Since the divisor and dividend are already written in descending order, we may begin to divide.

$$\begin{array}{r l} 2x - 7 & \\ 3x + 4\overline{)\,6x^2 - 13x + 13} & \\ \underline{{}^{-}6x^2 \overset{-}{\cancel{+}}\ 8x \quad \downarrow} & \text{Multiply } 2x \text{ by } 3x + 4 \text{ and subtract.} \\ -\ 21x + 13 & \\ \underline{\overset{+}{\cancel{-}}\ 21x \overset{+}{\cancel{-}}\ 28} & \text{Multiply } -7 \text{ by } 3x + 4 \text{ and subtract.} \\ 41 & \end{array}$$

Quick Check 7
Divide $\dfrac{8x^2 + 2x - 77}{4x - 9}$.

The remainder is 41. We write the quotient, and add a fraction with the remainder in the numerator and the divisor in the denominator.

$$\frac{6x^2 - 13x + 13}{3x + 4} = 2x - 7 + \frac{41}{3x + 4}$$

Using Placeholders When Dividing a Polynomial by a Polynomial

Objective 4 **Use placeholders when dividing a polynomial by a polynomial.**

Suppose that we wanted to divide $x^3 - 19x - 8$ by $x - 6$. Notice that the dividend is missing an x^2 term. When this is the case we add the term $0x^2$ as a **placeholder**. We add placeholders to dividends that are missing terms of a particular degree.

EXAMPLE 8 Divide $(x^3 - 19x - 8) \div (x - 6)$.

Solution

The degree of the dividend is 3, and each degree lower than 3 must be represented in the dividend. We will add the term $0x^2$ as a placeholder. The divisor does not have any missing terms.

$$
\begin{array}{r|l}
 & x^2 + 6x + 17 \\
\hline
x - 6 & x^3 + 0x^2 - 19x - 8 \\
 & {}^{-}x^3 \overset{+}{\cancel{-}} 6x^2 \quad \downarrow \quad \downarrow \\
\hline
 & \quad 6x^2 - 19x - 8 \\
 & \quad {}^{-}6x^2 \overset{+}{\cancel{-}} 36x \quad \downarrow \\
\hline
 & \qquad 17x - 8 \\
 & \qquad \overset{-}{\cancel{+}} 17x \overset{+}{\cancel{-}} 102 \\
\hline
 & \qquad\qquad 94
\end{array}
$$

Multiply x^2 by $x - 6$ and subtract.

Multiply $6x$ by $x - 6$ and subtract.

Multiply 17 by $x - 6$ and subtract.

$$(x^3 - 19x - 8) \div (x - 6) = x^2 + 6x + 17 + \frac{94}{x - 6}.$$

Quick Check 8

Divide $\dfrac{x^3 + 6x^2 + 43}{x + 9}$.

Building Your Study Strategy **Test Taking, 7** **Review Your Test** Try to leave yourself enough time to review the test at the end of the test period. Check for careless errors, which can cost you a significant number of points. Also be sure that your answers make sense within the context of the problem. For example, if the question asks how tall a person is and your answer is 68 feet tall, then chances are that something has gone astray.

Check for problems, or parts of problems, that you may have skipped and left blank. There is no worse feeling in a math class than getting a test back and realizing that you simply forgot to do one or more of the problems.

Take all of the allotted time to review your test. There is no reward for turning in a test early, and the longer that you spend on the test, the more likely it is that you will find a mistake or a problem where you can gain points. Keep in mind that a majority of the students who turn in tests early do so because they were not prepared for the test and cannot do several of the problems.

EXERCISES 5.7

Vocabulary

1. To divide a monomial by another monomial, divide the coefficients first and then divide the variables by ________ the exponents of like bases.

2. To divide a polynomial by a monomial, divide each ________ of the polynomial by the monomial.

3. If one polynomial divides evenly into another polynomial, with no remainder, then the first polynomial is said to be a(n) ________ of the second polynomial.

4. ________ are added to dividends that are missing terms of a particular degree.

Divide.

5. $\dfrac{42x^8}{6x^2}$

6. $\dfrac{56x^7}{7x^3}$

7. $\dfrac{-60x^8}{5x}$

8. $\dfrac{32x^{12}}{8x^{11}}$

9. $\dfrac{26x^7y^6}{2xy^5}$

10. $\dfrac{24x^5y^{12}}{4x^5y^4}$

11. $(120x^5) \div (10x^2)$

12. $(39x^{10}) \div (3x^5)$

13. $\dfrac{45x^9}{25x^7}$

14. $\dfrac{-56x^{15}}{12x^8}$

15. $\dfrac{3x^{14}}{36x^3}$

16. $\dfrac{7x^{21}}{42x^8}$

Find the missing monomial.

17. $\dfrac{?}{4x^3} = 3x^8$

18. $\dfrac{?}{7x^6} = -9x^{11}$

19. $\dfrac{32x^{10}}{?} = 8x^3$

20. $\dfrac{36x^{16}}{?} = \dfrac{6x^2}{5}$

Divide.

21. $\dfrac{10x^2 - 22x - 2}{2}$

22. $\dfrac{5x^2 + 45x}{5}$

23. $\dfrac{3x^4 + 39x^3 - 21x^2 + 9x}{3x}$

24. $\dfrac{28x^6 - 40x^3 - 4x^2}{4x}$

25. $(14x^8 + 21x^6 - 35x^3) \div (7x^2)$

26. $(-18x^{10} + 30x^9 - 6x^8) \div (6x^2)$

27. $\dfrac{48x^6 - 56x^5}{-8x^3}$

28. $\dfrac{-12x^5 + 8x^4 + 16x^3 - 4x^2}{-4x^2}$

29. $\dfrac{9x^5y^6 - 27x^2y^5 + 3x^3y^4}{3xy^4}$

30. $\dfrac{2a^8b^6 - 18a^5b^4 - 30a^2b^2}{2a^2b}$

Find the missing dividend or divisor.

31. $\dfrac{27x^3 - 45x^2 - 12x}{?} = 9x^2 - 15x - 4$

32. $\dfrac{4x^7 - 20x^5 + 36x^4}{?} = x^4 - 5x^2 + 9x$

33. $\dfrac{?}{6x^4} = x^2 - 7x - 19$

34. $\dfrac{?}{7x^2} = -x^4 - 8x^2 + 2$

Divide using long division.

35. $\dfrac{x^2 + 13x + 40}{x + 5}$

36. $\dfrac{x^2 + 15x + 36}{x + 3}$

37. $\dfrac{x^2 + x - 72}{x - 8}$

38. $\dfrac{x^2 + 5x - 14}{x + 7}$

39. $(x^2 - 21x + 90) \div (x - 6)$

40. $(x^2 - 15x + 77) \div (x - 9)$

41. $\dfrac{x^2 + 10x - 32}{x - 3}$

42. $\dfrac{x^2 + 14x + 37}{x + 5}$

43. $\dfrac{x^3 - 8x^2 - 21x + 9}{x + 2}$

44. $\dfrac{x^3 + 5x^2 - 13x - 32}{x + 4}$

45. $\dfrac{2x^2 + 19x + 30}{x + 8}$

46. $\dfrac{3x^2 - 13x - 75}{x - 7}$

47. $\dfrac{6x^2 - 17x - 75}{2x - 9}$

48. $\dfrac{4x^2 - 34x + 92}{2x - 5}$

49. $\dfrac{x^2 - 49}{x - 7}$

50. $(9x^2 - 169) \div (3x + 13)$

51. $\dfrac{x^4 + 2x^2 + 8x - 27}{x - 5}$

52. $\dfrac{x^5 - 9x^4 - 10x^2 + x + 17}{x + 2}$

53. $\dfrac{x^3 - 64}{x - 4}$

54. $\dfrac{x^3 + 512}{x + 8}$

Find the missing dividend or divisor.

55. $\dfrac{?}{x + 7} = x - 6$

56. $\dfrac{?}{x - 9} = x - 15$

57. $\dfrac{x^2 - 14x - 72}{?} = x + 4$

58. $\dfrac{6x^2 - 5x - 99}{?} = 2x - 9$

59. Is $x + 11$ a factor of $x^2 + 34x + 253$?

60. Is $x - 3$ a factor of $x^2 + 11x - 39$?

61. Is $2x + 7$ a factor of $6x^2 + x - 63$?

62. Is $5x + 8$ a factor of $45x^2 + 37x - 56$?

Mixed Practice, 63–80

Divide.

63. $\dfrac{x^2 - 11x + 20}{x - 4}$

64. $\dfrac{x^2 + 6x - 7}{x + 9}$

65. $\dfrac{x^3 + 2x^2 - 7}{x + 6}$

66. $\dfrac{24x^{10}}{4x^2}$

67. $\dfrac{9x^6 - 12x^5 + 21x^4 - 3x^3}{3x^3}$

68. $\dfrac{x^2 - x - 210}{x + 14}$

69. $\dfrac{4x^2 + 19x - 48}{x + 7}$

70. $\dfrac{3x^2 - 9x - 51}{x - 5}$

71. $\dfrac{20x^2 - 37x + 7}{4x - 1}$

72. $\dfrac{8x^2 - 26x + 27}{2x - 3}$

73. $\dfrac{8x^8 - 6x^6 - 18x^5 + 4x^3}{-2x^3}$

74. $\dfrac{6x^2 + 5x - 391}{3x - 23}$

75. $\dfrac{8x^3 - 26x - 9}{4x + 6}$

76. $\dfrac{72a^9b^{10}c^{11}}{6ab^3c^{11}}$

77. $\dfrac{6x^3 - 37x^2 + 36x + 70}{2x - 7}$

78. $\dfrac{30x^{10} - 42x^7 - 57x^4}{3x^4}$

79. $\dfrac{-54x^{10}y^4z^8}{6x^7y^3z}$

80. $\dfrac{x^5 + 242}{x + 3}$

Writing in Mathematics

Answer in complete sentences.

81. Explain the use of placeholders when dividing a polynomial by another polynomial.

82. ***Solutions Manual**** Write a solutions manual page for the following problem:

Divide $\dfrac{6x^2 - 17x - 19}{2x - 7}$.

83. ***Newsletter**** Write a newsletter that explains how to divide a polynomial by another polynomial using long division.

***See Appendix B for details and sample answers.**

Chapter 5 Summary

Section 5.1—Topic	Chapter Review Exercises
Evaluating Rational Expressions	1–4
Finding Values for Which a Rational Expression Is Undefined	5–6
Simplifying Rational Expressions to Lowest Terms	7–10
Evaluating Rational Functions	11–12
Finding the Domain of a Rational Function	13–14
Section 5.2—Topic	**Chapter Review Exercises**
Multiplying Two Rational Expressions	15–18
Multiplying Two Rational Functions	19–20
Dividing a Rational Expression by Another Rational Expression	21–24
Dividing a Rational Function by Another Rational Function	25–26
Section 5.3—Topic	**Chapter Review Exercises**
Adding or Subtracting Rational Expressions with the same Denominator	27–29
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Adding or Subtracting Rational Expressions with Unlike Denominators	33–37
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Section 5.4—Topic	**Chapter Review Exercises**
Simplifying Complex Numerical Fractions	40–43
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Section 5.6—Topic	**Chapter Review Exercises**
Solving Applied Problems Involving Rational Equations	55–64
Section 5.7—Topic	**Chapter Review Exercises**
Dividing Polynomials	65–72

Summary of Chapter 5 Study Strategies

Attending class each day and doing all of your homework does not guarantee a good grade when taking an exam. Through careful preparation and the adoption of the test-taking strategies introduced in this chapter, you can maximize your grade on a math exam. Here is a summary of the points that have been introduced.

- Make the most of your time before the exam.
- Write down important facts as soon as you get your test.
- Briefly read through the test.
- Begin by solving the easier problems first.
- Review your test as thoroughly as possible before turning it in.

Evaluate the rational expression for the given value of the variable. [5.1]

1. $\dfrac{9}{x-5}$ for $x = -7$

2. $\dfrac{x-4}{x+20}$ for $x = 16$

3. $\dfrac{x^2+7x-8}{x^2-2x+6}$ for $x = 6$

4. $\dfrac{x^2-3x+12}{x^2+5x-4}$ for $x = -8$

Find all values for which the rational expression is undefined. [5.1]

5. $\dfrac{9}{2x-3}$

6. $\dfrac{x^2+15x+56}{x^2+3x-28}$

Simplify the given rational expression. (Assume that all denominators are nonzero.) [5.1]

7. $\dfrac{x-8}{x^2-64}$

8. $\dfrac{x^2+7x-18}{x^2-7x+10}$

9. $\dfrac{4x-x^2}{x^2-11x+28}$

10. $\dfrac{2x^2+11x-6}{x^2+2x-24}$

Evaluate the given rational function. [5.1]

11. $r(x) = \dfrac{x-9}{x^2-5x+16}, r(-3)$

12. $r(x) = \dfrac{x^2+3x+14}{x^2-5x}, r(7)$

Find the domain of the given rational function. [5.1]

13. $r(x) = \dfrac{x^2+11x+32}{x^2-9x}$

14. $r(x) = \dfrac{x^2+3x-25}{x^2-5x-6}$

Multiply. [5.2]

15. $\dfrac{x+8}{x-3} \cdot \dfrac{x^2-13x+30}{x^2+14x+48}$

16. $\dfrac{x^2+16x+63}{x^2+x-12} \cdot \dfrac{x^2-16}{x^2+3x-54}$

17. $\dfrac{x^2+5x}{3x^2+4x-4} \cdot \dfrac{x^2+4x+4}{x^2-4x-45}$

18. $\dfrac{49-x^2}{x+6} \cdot \dfrac{x^2+5x-6}{x^2-4x-21}$

For the given functions $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$. [5.2]

19. $f(x) = \dfrac{x+8}{x^2+2x}, g(x) = \dfrac{x^2+12x+20}{x^2+2x-80}$

20. $f(x) = \dfrac{x^2+x-30}{x^2+2x-99}, g(x) = \dfrac{x^2-5x-36}{x^2-x-20}$

Divide. [5.2]

21. $\dfrac{x^2+8x+7}{x^2-3x-40} \div \dfrac{x+1}{x+5}$

22. $\dfrac{x^2+10x-24}{x^2+10x+24} \div \dfrac{x^2-5x+6}{x^2+13x+42}$

23. $\dfrac{5x^2+7x-6}{x^2+8x} \div \dfrac{x^2-3x-10}{x^2-64}$

24. $\dfrac{100-x^2}{x-7} \div \dfrac{x^2-13x+30}{x^2-x-42}$

For the given functions $f(x)$ and $g(x)$, find $f(x) \div g(x)$. [5.2]

25. $f(x) = \dfrac{x^2+3x}{x^2+8x+16}, g(x) = \dfrac{x^2+10x+21}{x^2-3x-28}$

26. $f(x) = \dfrac{x^2+15x-16}{x^2+9x+14}, g(x) = \dfrac{x^2-6x+5}{x^2-4x-12}$

Add or subtract. [5.3]

27. $\dfrac{15}{x+4} + \dfrac{9}{x+4}$

28. $\dfrac{x^2+4x}{x^2+11x+24} - \dfrac{7x+18}{x^2+11x+24}$

29. $\dfrac{x^2-6x-8}{x^2+4x-45} + \dfrac{6x-17}{x^2+4x-45}$

30. $\dfrac{4x+9}{x-3} - \dfrac{2x-27}{3-x}$

Worked-out solutions to Review Exercises marked with [circle] can be found on page AN-2.

31. $\dfrac{2x^2 + 3x + 21}{x - 2} + \dfrac{x^2 + 13x + 5}{2 - x}$

32. $\dfrac{x^2 + 6x + 10}{x^2 - 16} - \dfrac{8x + 30}{16 - x^2}$

33. $\dfrac{5}{x^2 - 3x - 4} + \dfrac{4}{x^2 - 12x + 32}$

34. $\dfrac{9}{x^2 + 13x + 22} - \dfrac{2}{x^2 + 20x + 99}$

35. $\dfrac{x}{x^2 - 16} - \dfrac{5}{x^2 + 2x - 24}$

36. $\dfrac{x + 6}{x^2 + 18x + 80} - \dfrac{2}{x^2 + 14x + 48}$

37. $\dfrac{x + 5}{x^2 + 4x + 3} + \dfrac{x - 5}{x^2 + 5x + 4}$

For the given rational functions $f(x)$ and $g(x)$, find $f(x) - g(x)$. **[5.3]**

38. $f(x) = \dfrac{3}{x^2 - 4x - 5}, g(x) = \dfrac{1}{x^2 - 12x + 35}$

For the given rational functions $f(x)$ and $g(x)$, find $f(x) - g(x)$. **[5.3]**

39. $f(x) = \dfrac{x - 9}{x^2 - 17x + 70}, g(x) = \dfrac{4}{x^2 - 8x - 20}$

Simplify the complex fraction. **[5.4]**

40. $\dfrac{1 + \dfrac{6}{x}}{1 - \dfrac{36}{x^2}}$

41. $\dfrac{\dfrac{x + 8}{x - 4} - \dfrac{2}{x}}{\dfrac{x^2 + 9x + 20}{x^2 - 4x}}$

42. $\dfrac{1 + \dfrac{19}{x} + \dfrac{90}{x^2}}{1 + \dfrac{8}{x} - \dfrac{9}{x^2}}$

43. $\dfrac{\dfrac{x^2 + 5x - 14}{x^2 - 7x + 12}}{\dfrac{x^2 + 13x + 42}{x^2 - 16}}$

Solve. **[5.5]**

44. $\dfrac{10}{x} - \dfrac{1}{6} = \dfrac{11}{24}$

45. $x + 6 - \dfrac{48}{x} = 4$

46. $\dfrac{2}{x - 10} = \dfrac{7}{x + 15}$

47. $\dfrac{2x + 3}{x + 3} = \dfrac{x - 2}{x - 5}$

48. $\dfrac{5}{x - 4} + \dfrac{4}{x + 3} = \dfrac{x^2 - 3x + 31}{x^2 - x - 12}$

49. $\dfrac{x}{x + 6} + \dfrac{7}{x - 8} = \dfrac{84}{x^2 - 2x - 48}$

50. $\dfrac{x - 3}{x^2 - x - 30} + \dfrac{4}{x^2 + x - 20} = \dfrac{3}{x^2 - 10x + 24}$

51. $\dfrac{6}{x^2 + 7x - 8} - \dfrac{2}{x^2 + 2x - 3} = \dfrac{x + 4}{x^2 + 11x + 24}$

Solve for the specified variable. **[5.5]**

52. $\dfrac{b}{2} = \dfrac{A}{h}$ for h

53. $y = \dfrac{5x}{2x - 3}$ for x

54. $\dfrac{x}{3r} - \dfrac{y}{4r} = \dfrac{1}{6}$ for r

55. The sum of the reciprocal of a number and $\dfrac{7}{12}$ is $\dfrac{3}{4}$. Find the number. [5.6]

56. One positive number is 12 more than another positive number. If four times the reciprocal of the smaller number is added to six times the reciprocal of the larger number, the sum is equal to 1. Find the two numbers. [5.6]

57. Jerry can mow a lawn in 40 minutes, while George takes 50 minutes to mow the same lawn. If Jerry and George work together, using two lawn mowers, how long would it take them to mow the lawn? [5.6]

58. Two pipes, working together, can fill a tank in 10 hours. Working alone, it would take the smaller pipe 15 hours longer than it would take the larger pipe to fill the tank. How long would it take the smaller pipe alone to fill the tank? [5.6]

59. Margaret had to make a 305-mile drive to Atlanta. After driving the first 110 miles, she increased her speed by 10 miles per hour. If the drive took her exactly 5 hours, find the speed at which she was driving for the first 110 miles. [5.6]

60. Sarah is kayaking in a river that is flowing downstream at a speed of 2 miles per hour. Sarah paddled 4 miles upstream, then turned around and paddled 8 miles downstream. This took Sarah a total of 2 hours. What is the speed that Sarah can paddle in still water? [5.6]

61. The number of calories in a glass of soda varies directly as the amount of soda. If a 12-ounce serving of soda has 180 calories, how many calories are there in an 8-ounce glass of soda? [5.6]

62. The distance required for a car to stop after applying the brakes varies directly as the square of the speed of the car. If it takes 125 feet for a car traveling at 50 miles per hour to stop, how far would it take for a car traveling 80 miles per hour to come to a stop? [5.6]

63. The maximum load that a wooden beam can support varies inversely as its length. If a beam that is 8 feet long can support 725 pounds, what is the maximum load that can be supported by a beam that is 10 feet long? [5.6]

64. The illumination of an object varies inversely as the square of its distance from the source of light. If a light source provides an illumination of 30 foot-candles at a distance of 10 feet, find the illumination at a distance of 5 feet. [5.6]

Divide. **[5.7]**

65. $\dfrac{42x^9}{7x^7}$

66. $\dfrac{24x^5 - 40x^4 - 88x^3 + 8x^2}{8x^2}$

67. $\dfrac{x^2 + 21x - 130}{x - 5}$

68. $\dfrac{x^2 - 5x - 113}{x + 8}$

69. $\dfrac{6x^2 + 11x - 1}{x + 3}$

70. $\dfrac{8x^3 - 38x^2 + 13x + 80}{2x - 5}$

71. $\dfrac{x^3 + 7x - 20}{x - 6}$

72. $\dfrac{x^3 - 515}{x - 8}$

Chapter 5 Test

For Extra Help — Pass the Test — Test solutions are found on the enclosed CD.

Evaluate the rational expression for the given value of the variable.

1. $\dfrac{8}{x-6}$ for $x = -12$

Find all values for which the rational expression is undefined.

2. $\dfrac{x^2+13x+42}{x^2-13x+40}$

Simplify the given rational expression. (Assume that all denominators are nonzero.)

3. $\dfrac{x^2-13x+36}{x^2-2x-63}$

Evaluate the rational function.

4. $r(x) = \dfrac{x^2+6x-20}{x^2-2x+24}, r(-4)$

Multiply.

5. $\dfrac{x^2+7x-8}{x^2+7x+12} \cdot \dfrac{x^2-7x-30}{x^2-6x+5}$

Divide.

6. $\dfrac{x^2-6x+9}{x^2+6x+8} \div \dfrac{9-x^2}{x^2-8x-20}$

For the given functions $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$.

7. $f(x) = \dfrac{x^2+x-42}{x^2-2x}, g(x) = \dfrac{x-2}{x^2+2x-35}$

Add or subtract.

8. $\dfrac{5x-3}{2x+8} - \dfrac{2x-15}{2x+8}$

9. $\dfrac{2x^2+8x+13}{x-5} + \dfrac{x^2+4x+58}{5-x}$

10. $\dfrac{x-5}{x^2-4x+3} - \dfrac{6}{x^2+x-2}$

11. $\dfrac{3}{x^2-16x+63} + \dfrac{6}{x^2-10x+21}$

For the given rational functions $f(x)$ and $g(x)$, find $f(x) + g(x)$.

12. $f(x) = \dfrac{x}{x^2+9x+18}, g(x) = \dfrac{5}{x^2+x-6}$

Simplify the complex fraction.

13. $\dfrac{1 - \dfrac{4}{x} - \dfrac{12}{x^2}}{1 - \dfrac{15}{x} + \dfrac{54}{x^2}}$

Solve.

14. $\dfrac{14}{x} + \dfrac{4}{15} = \dfrac{29}{30}$

15. $\dfrac{x+6}{x^2+6x+8} - \dfrac{4}{x^2-6x-16} = \dfrac{3}{x^2-4x-32}$

Solve for the specified variable.

16. $x = \dfrac{4y}{y+8}$ for y

17. Two pipes, working together, can fill a tank in 60 minutes. Working alone, it would take the smaller pipe 90 minutes longer than it would take the larger pipe to fill the tank. How long would it take the smaller pipe alone to fill the tank?

18. Preparing for a race, Lance rode his bicycle 23 miles. After the first 8 miles, he increased his speed by 5 miles per hour. If the ride took him exactly 1 hour, find the speed at which he was riding for the first 8 miles.

Divide.

19. $\dfrac{2x^2-13x-99}{x+4}$

20. $\dfrac{2x^3-9x^2-109}{x-7}$

Mathematicians in History

John Nash

John Nash is an American mathematician whose research has greatly affected mathematics and economics, as well as many other fields. When Nash was applying to graduate school at Princeton, one of his math professors said simply, "This man is a genius."

Write a one-page summary (OR make a poster) of the life of John Nash and his accomplishments.

Interesting issues:

- Where and when was John Nash born?
- Describe Nash's childhood, as well as his life as a college student.
- What mental illness struck Nash in the late 1950's?
- Nash was an influential figure in the field of game theory. What is game theory?
- In 1994, Nash won the Nobel Prize for Economics. Exactly what did Nash win the prize for?
- One of Nash's nicknames is "The Phantom of Fine Hall." Why was this nickname chosen for him?
- What color sneakers did Nash wear?
- Sylvia Nasar wrote a biography of Nash's life, which was made into an Academy Award–winning movie. What was the title of the book and movie?
- What actor played John Nash in the movie?

The weight W of an object varies inversely as the square of the distance d from the center of the Earth. At sea level (3978 miles from the center of the Earth) a person weighs 150 pounds. The formula used to compute the weight of this person at different distances from the center of the Earth is $W = \frac{2{,}373{,}672{,}600}{d^2}$.

a) Use the given formula to calculate the weight of this person at different distances from the center of the Earth. Round to the nearest tenth of a pound.

	Distance from center of Earth	Weight
On top of the world's tallest building, Taipei 101 (Taipei, Taiwan)	3978.316 miles	
On top of the world's tallest structure, KVLY-TV mast (Mayville, ND)	3978.391 miles	
In an airplane	3984.629 miles	
In the space station	4201.402 miles	
Halfway to the moon	123,406 miles	
Halfway to Mars	4,925,728 miles	

b) What do you notice about the weight of the person as the distance from the center of the Earth increases?

c) What do you think is the reason for what you've observed in b)?

d) As accurately as possible, plot these points on an axis system in which the horizontal axis represents the distance and the vertical axis represents the weight. Do the points support your observations from b)?

e) How far from the center of the Earth would this 150-pound person have to travel to weigh 100 pounds?

f) How far from the center of the Earth would this 150-pound person have to travel to weigh 75 pounds?

g) How far from the center of the Earth would this 150-pound person have to travel to weigh 0 pounds?